Algebraic Geometry II - Summary of my summary

1 Cohomology of q.c. sheaves of modules

1.1. Recollection of basic definitions and results

Prop 1.1.1: Intersection $U \cap V$ of affines U, V in a scheme X is again affine.

Prop 1.1.2: If $\mathcal{M}$ q.c. $\mathcal{O}_X$-module, $f: X \to Y$ q.c. and q.s., then $f_*\mathcal{M}$ q.c. $\mathcal{O}_Y$-module

Prop 1.1.3: $Qc(X)$ is closed under kernels, cokernels and direct sums.

Prop 1.1.4: Closed subpreschemes of X correspond to q.c. ideals of $\mathcal{O}_X$.

Cor 1.1.1: For $f: \mathcal{M} \to \mathcal{N}$ of q.c. $\mathcal{O}_X$-modules, $\ker(f)$ and $\operatorname{coker}(f)$ give the expected thing on affine open subsets.

Cor 1.1.2: Taking local sections on open affine $U \subseteq X$ is exact functor on Qc. $\mathcal{O}_X$-modules.

1.2 Čech cohomology

Def 1.2.1: The Čech complex $\check{C}^*(\mathcal{U}, \mathcal{M})$ for open cover $\mathcal{U}: X = \bigcup_{i \in I} U_i$, q.c. $\mathcal{O}_X$-module $\mathcal{M}$

Prop 1.2.1: For X a scheme and $\mathcal{U}$ affine open cover, a short exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ of q.c. $\mathcal{O}_X$-modules gives l.e.s.

$$\cdots \to \check{H}^{i-1}(\mathcal{U}, \mathcal{M}'') \to \check{H}^i(\mathcal{U}, \mathcal{M}') \to \check{H}^i(\mathcal{U}, \mathcal{M}) \to \check{H}^i(\mathcal{U}, \mathcal{M}'') \to \check{H}^i(\mathcal{U}, \mathcal{M}') \to \cdots$$

Lemma 1.2.2: For $f: X \to Y$, $\check{C}^*(f^{-1}\mathcal{U}, \mathcal{M}) \cong \check{C}^*(\mathcal{U}, f_*\mathcal{M})$.

Theorem 1: a) For any affine open cover $\mathcal{U}$ of q.c. scheme X

$$H^*(X, \mathcal{M}) \xrightarrow{\cong} \check{H}^*(\mathcal{U}, \mathcal{M}) \xleftarrow{\cong} {}^{a}\check{H}^*(\mathcal{U}, \mathcal{M})$$

b) When X affine, $H^i(X, \mathcal{M}) = 0$ for $i > 0$.

c) Canonical isomorphism $H^0(X, \mathcal{M}) \cong \mathcal{M}(X)$

d) For s.e.s. $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ we have l.e.s as above.

Cor 1.2.2 For s.e.s. $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ on any prescheme X, if two of the three are q.c., so is the third, and taking local sections in affine U keeps exactness

Prop 1.2.3: For q.c. scheme X, $\mathcal{M} \in Qc(X)$, $f \in \mathcal{O}_X(X)$,

$$H^*(X, \mathcal{M})_f \xrightarrow{\cong} H^*(X \setminus V(f), \mathcal{M}|_{X \setminus V(f)})$$

1.3 The affinity criterion of Serre

Prop 1.3.1: For a q.c. scheme X, t.f.a.e.:

a) X is affine

b) For all q.c. sheaves of $\mathcal{O}_X$-modules $\mathcal{M}$ and all $p > 0$, $H^p(X, \mathcal{M}) = 0$.

c) For any q.c. sheaf of ideals $\mathcal{J}$ on X, $H^1(X, \mathcal{J}) = 0$.

1.4 Cohomological dimension

Prop 1.4.1: Let X be a scheme, $Z \subseteq X$ a Noetherian closed subset, and $\mathcal{M}$ a q.c. $\mathcal{O}_X$-module st. $\mathcal{M}|_{X \setminus Z} = 0$. Then $H^p(X, \mathcal{M}) = 0$ for $p > \dim(Z)$.

1.5 Higher direct images

Def 1.5.1: For q.c. separated $f: X \to Y$, $R^p f_*\mathcal{M}$ is sheaf. of $U \mapsto H^p(f^{-1}(U), \mathcal{M})$.

Prop 1.5.1 a) $R^0 f_*\mathcal{M} \cong f_*\mathcal{M}$ b) The $R^p f_*\mathcal{M}$ are q.c. $\mathcal{O}_Y$-modules

c) Long exact sequence $\cdots \to R^p f_*\mathcal{M} \to R^p f_*\mathcal{M}'' \to R^{p+1} f_*\mathcal{M}' \to R^{p+1} f_*\mathcal{M} \to \cdots$

d) For U affine, canonical map $H^p(f^{-1}(U), \mathcal{M}) \xrightarrow{\cong} R^p f_*\mathcal{M}(U)$ is iso.

Def 1.5.2: We have a "pull-back morphism" $H^p(Y, f_*\mathcal{M}) \to H^p(X, \mathcal{M})$.

1.6 Affine morphisms

Prop 1.6.1: For a q.c. q.s. morphism with X q.s. t.f.a.e:

a) For all open affine $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is affine.

b) Y can be covered by such U.

c) f is separated and for all q.c. $\mathcal{O}_X$-modules $\mathcal{M}$ and $p > 0$, $R^p f_*\mathcal{M} = 0$.

Cor 1.6.1: Let $f: X \to Y$ be affine and M a q.c. $\mathcal{O}_X$-module. Then $R^p f_* M = 0$ for $p > 0$ and $H^p(Y, f_* M) \cong H^p(X, M)$.

Prop 1.6.2: Let A be a q.c. $\mathcal{O}_Y$-algebra
a) The morphism $\underline{\mathrm{Spec}}(A) \longrightarrow Y$ is affine
b) For $\xi: X \to Y$, there is a bijection
$$\mathrm{Hom}_{Y\text{-ps}}(X, \underline{\mathrm{Spec}}\, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_Y\text{-alg}}(A, \xi_* \mathcal{O}_X)$$
c) $\xi: X \to Y$ is affine iff $\xi_* \mathcal{O}_X$ is q.c. $\mathcal{O}_Y$-module and $X \xrightarrow{\cong} \underline{\mathrm{Spec}}\, \xi_* \mathcal{O}_X$.

Cor 1.6.2: $\underline{\mathrm{Spec}}(f^* A) \cong \underline{\mathrm{Spec}}(A) \times_Y \tilde{Y}$ for $f: \tilde{Y} \to Y$.

1.7 The relation between H^1 and torseurs

Def 1.7.1: A G-torseur for a sheaf of groups G

Ex 1.7.2: Line bundles correspond with $\mathcal{O}_X^*$-torseurs.

Prop 1.7.1: Bijection between $\check{H}^1(\mathfrak{U}, G)$ and G-torseurs trivial on $\mathfrak{U}$.

Def 1.7.2: The morphism $\mathrm{dlog}: \mathcal{O}_X^* \longrightarrow \Omega_{X/S}$; $f \longmapsto df/f$ gives c_1 by:
$$\mathrm{Pic}(X) \cong \{\mathcal{O}_X^*\text{-torseurs}\}/\sim \; \cong H^1(X, \mathcal{O}_X^*) \xrightarrow{(\mathrm{dlog})_*} H^1(X, \Omega_{X/S})$$

2. Cohomology of projective spaces

2.1 Regular sequences and the Koszul complex

Def 2.1.1: M-regular sequence $(x_0, \dots, x_n) \in R^{n+1}$

Def 2.1.2: Koszul complex $C^*(\underline{x}, M)$

Fact 2.1.1: Regularity conditions for $\underline{x} = (x_0, \dots, x_n)$

Cor 2.1.1: Let $S := R[X_0, \dots, X_n]$, $\underline{X}^q := (X_0^q, \dots, X_n^q)$, which is regular. Then
a) $H^k(C^*(\underline{X}^q, S)) = 0$ for $k \neq n+1$
b) $H^{n+1}(C^*(\underline{X}^q, S))_k = 0$ for $k > -n-1$
c) $H^{n+1}(C^*(\underline{X}^q, S))_{-n-1} \cong R$
d) There is a nondegenerate pairing
$$S_k \times H^{n+1}(\underline{X}^q, S)_{-n-1-k} \longrightarrow H^{n+1}(\underline{X}^q, S)_{-n-1} \xrightarrow{\cong} R.$$

2.2 The scheme $\mathbb{P}^n_A$ and its cohomology

Prop 2.2.1: Let $R_\cdot$ be an $\mathbb{N}$-graded ring.
a) $\mathrm{Proj}(R_\cdot)$ is a prescheme with affine topology base $\{\mathrm{Proj}(R_\cdot) \setminus V(f) \mid f \in R_k, k > 0\}$
b) Every finite subset contained in affine set (so $\mathrm{Proj}(R_\cdot)$ is a scheme)
c) Sheaves of modules $\mathcal{O}(k)$ are q.c.: $\mathcal{O}(k)|_{\mathrm{Proj}(R_\cdot) \setminus V(f)} \cong \widetilde{(R_f)_k}$
d) When $R_\cdot$ generated by R_0, R_1, the $\mathcal{O}(k)$ are line bundles: $\mathcal{O}(k) \otimes \mathcal{O}(\ell) \cong \mathcal{O}(k+\ell)$.
e) We have $\pi: \mathrm{Proj}(R_\cdot) \longrightarrow \mathrm{Spec}(R_0)$, which is of f.t. if R/R_0 is.
f) If $R_\cdot$ is Noetherian, $\mathrm{Proj}(R_\cdot)$ is Noetherian.

Theorem 2: Let $R := A[X_0, \dots, X_n]$, $X := \mathbb{P}^n_A = \mathrm{Proj}(R_\cdot)$
a) For all k, $R_k \xrightarrow{\cong} \mathcal{O}(k)(X) = H^0(X, \mathcal{O}(k))$
b) For $0 < p < n$, all k, $H^p(X, \mathcal{O}(k)) = 0$
c) For $p = n > 0$, $H^n(X, \mathcal{O}(k)) = 0$ for $k > -n-1$
d) For $p = n$, $k = -n-1$, $H^n(X, \mathcal{O}(-n-1)) \cong A$
e) There is a non-degenerate pairing
$$H^0(X, \mathcal{O}(k)) \times H^n(X, \mathcal{O}(-n-1-k)) \longrightarrow H^n(X, \mathcal{O}(-n-1)) \cong A.$$

Def 2.2.1: Ample line bundle

Lemma 2.2.2: Let X be q.c., $\mathcal{L}$ a line bundle with $\lambda \in \mathcal{L}(X)$, M a q.c. $\mathcal{O}_X$-module.
a) If $m \in M(X)$ has $m|_{X \setminus V(\lambda)} = 0$, then $\lambda^{\otimes k} \otimes m = 0$ in $(\mathcal{L}^{\otimes k} \otimes M)(X)$ for some k
b) If X also q.s., and $m \in M(X \setminus V(\lambda))$, then some $\lambda^{\otimes k} \otimes m$ extends to all of X.

Prop 2.2.2: For X q.c. and q.s., if there are $s_1, \dots, s_n \in \mathcal{L}(X)$ s.t. $U_i := X \setminus V(s_i)$ is affine and cover X, then $\mathcal{L}$ is ample.

Cor 2.2.1: $\mathcal{O}(1)$ on $\mathbb{P}^n_A$ is ample.

Theorem 3: Let A be a Noetherian ring, $X = \mathbb{P}^n_A$, M a ~~q.c.~~ coh. $\mathcal{O}_X$-module. Then the groups $H^p(X, M)$ are f.g. A-modules and for $p > 0$, $H^p(X, M(k)) = 0$ for $k \gg 0$.

Prop 2.2.3: Let R be a Noetherian ring, $X \subseteq \mathbb{P}^n_R$ a closed subprescheme. Then for sufficiently large k we have a surjective map $H^0(\mathbb{P}^n_R, \mathcal{O}(k)) \to H^0(X, \mathcal{O}(k))$.

Prop 2.2.4: Let A be Noetherian, $R = A[X_0, \dots, X_n]$, M f.g. R-module, $X = \mathbb{P}^n_A$, $\mathcal{M} = \tilde{M}$.
a) When k is sufficiently large, $M_k \to H^0(X, \mathcal{M}(k))$ is an iso
b) If A is a field, then $\chi(X, \mathcal{M}(k)) = P_M(k)$ is the Hilbert polynomial of M

2.3 Projective morphisms

For $\mathcal{R}_\cdot = \oplus \mathcal{R}_i$ a graded q.c. $\mathcal{O}_X$-algebra, we defined $\underline{\mathrm{Proj}}(\mathcal{R}_\cdot) \xrightarrow{\pi} X$ by gluing for each open affine U_i: $\mathrm{Proj}(\mathcal{R}(U_i)_\cdot) \to \mathrm{Spec}(\mathcal{R}(U_i)_0) \to \mathrm{Spec}(\mathcal{O}_X(U_i)) = U_i$

Fact 2.3.1: a) $\underline{\mathrm{Proj}}(\mathcal{R}_\cdot)$ is a prescheme over X.
b) π is separated. For all affine open $U \subseteq X$, $\pi^{-1}(U)$ is a scheme.
c) When $\mathcal{R}_\cdot$ generated by $\mathcal{R}_0$, $\mathcal{R}_1$, the $\mathcal{O}(l)$ are line bundles behaving nicely.

Prop 2.3.1: Let $X \xrightarrow{\xi} S$ be an S-prescheme, $\mathcal{L}$ line bundle on X, $\mathcal{R}_\cdot$ a q.c. graded $\mathcal{O}_S$-algebra gen. by $\mathcal{R}_0$ and $\mathcal{R}_1$. Then we have the "adjunction"

$$\left\{ \begin{matrix} \text{Morphisms } \varphi: X \to P = \underline{\mathrm{Proj}}(\mathcal{R}_\cdot) \\ \text{with trivialization } \tau: \varphi^* \mathcal{O}(1)_P \xrightarrow{\cong} \mathcal{L} \end{matrix} \right\} \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{Graded } \alpha: \xi^* \mathcal{R}_\cdot \to \bigoplus_k \mathcal{L}^{\otimes k} \\ \text{of } \mathcal{O}_X\text{-algebras surjective on stalks} \end{matrix} \right\}$$

Cor 2.3.2: As above, for $T \xrightarrow{\eta} S$, we get

$$\underline{\mathrm{Proj}}(\eta^* \mathcal{R}_\cdot) \xrightarrow{\cong} (\underline{\mathrm{Proj}}(\mathcal{R}_\cdot)) \times_S T$$

Cor 2.3.3: For a q.c. $\mathcal{O}_S$-algebra $\mathcal{R}_\cdot$ and line bundle $\mathcal{L}$ on S set

$$\mathcal{R}_\cdot^{(\mathcal{L})} = \bigoplus_k \mathcal{R}_k \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes k}$$

Then $\underline{\mathrm{Proj}}(\mathcal{R}_\cdot^{(\mathcal{L})}) \xrightarrow{\cong} \underline{\mathrm{Proj}}(\mathcal{R}_\cdot)$.

Prop 2.3.2: For $\mathcal{O}_S$-algebras $\mathcal{R}_\cdot$, $\mathcal{S}_\cdot$ as before,

$$\underline{\mathrm{Proj}}(\mathcal{R}_\cdot) \times_S \underline{\mathrm{Proj}}(\mathcal{S}_\cdot) \xrightarrow{\cong} \underline{\mathrm{Proj}}(\bigoplus_k \mathcal{R}_k \otimes \mathcal{S}_k).$$

Ex 2.3.1 We have a closed embedding $\mathbb{P}^n_R \times_R \mathbb{P}^m_R \to \mathbb{P}^{n+m+nm}_R$

Def 2.3.1: We call $X \xrightarrow{\xi} S$ projective if $X \cong \underline{\mathrm{Proj}}(\mathcal{R}_\cdot)$ ($\mathcal{R}_\cdot$ loc. f.g.)
We call it strongly projective if it factors as $X \xrightarrow{i} \mathbb{P}^n_S \to S$ (i closed emb.)

Fact 2.3.2: Base change preserves (strong) projectivity.

Prop 2.3.3: For $i: X \to Y$ closed immersion, if $f: Y \to S$ is (str.) pr., so is $f \circ i$.

Cor 2.3.4: a) Every strongly projective morphism is projective.
b) If S has an ample line bundle, the converse also holds.

Prop 2.3.4: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are strongly projective, then gf is.
If f, g are projective and Z is q.c. and q.s., then gf is projective.

Cor 2.3.5: Finite morphisms are projective.

Theorem 4: Let $f: X \to S$ be projective, X, S locally Noetherian preschemes and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. Then each $R^k f_* \mathcal{M}$ is coherent.

Proposition 2.3.5: For $X \xrightarrow{f} Y \xrightarrow{g} S$, if g separated and gf (strongly) projective, then f is strongly projective.

~~Def 2.4.1~~ 2.4 Proper morphisms

Def 2.4.1: Proper morphism

Prop 2.4.1: The class of proper morphisms behaves 'nicely'.

Prop 2.4.2: The morphism $\underline{\mathrm{Proj}}(\mathcal{R}_\cdot) \to X$ is proper for loc. f.g. $\mathcal{O}_X$-algebra $\mathcal{R}_\cdot$.

Theorem 5: Let $X \xrightarrow{f} Y$ be a proper morphism, X, Y locally Noetherian.
If $\mathcal{M}$ is a ~~coh~~ coherent $\mathcal{O}_X$-module, then all $R^p f_* \mathcal{M}$ are coherent $\mathcal{O}_Y$-modules.

Cor 2.4.1: If A is Noetherian, X a proper Spec A-scheme and M a coherent $\mathcal{O}_X$-module, then each $H^p(X,M)$ is a f.g. A-module.

Theorem 6: Let $f: X \to \operatorname{Spec} A$ be proper. For a line bundle $\mathcal{L}$ on X t.f.a.e.
(a) $\mathcal{L}$ is ample
(b) Some power $\mathcal{L}^{\otimes k}$ is very ample.
(c) ...
(d) If M is a coherent $\mathcal{O}_X$-module, $p>0$, then for k large enough $H^p(X, M \otimes \mathcal{L}^{\otimes k}) = 0$

3. Cohomology of curves

For a divisor $D \in \operatorname{Div}(C)$ on a q.c. regular curve C we defined the line bundle

$$\mathcal{O}_C(D)(U) = \{ f \in K := \mathcal{O}_{C,\eta} \mid v_x(f) \geq -D(x) \ \forall x \in U \cap C_1 \}$$

3.1 Formulation of the results

Let k be an algebraically closed field. All schemes are over Spec k.

Theorem 7 (Serre Duality): Let $C \to \operatorname{Spec} k$ be a proper regular curve.
(a) There is a homomorphism ~~with~~ $\deg: \operatorname{Pic}(C) \to \mathbb{Z}$ s.t. $\deg(\mathcal{O}_C(D)) = \deg(D)$.
(b) There is an iso $\tau: H^1(C, \Omega_C) \xrightarrow{\cong} k$ s.t.

$$\begin{tikzcd} \operatorname{Pic}(C) \arrow[r, "c_1"] \arrow[d, "\deg"] & H^1(C, \Omega_C) \arrow[d, "\tau", "\cong"'] \\ \mathbb{Z} \arrow[r, "\cdot 1"] & k \end{tikzcd}$$

(c) For every vector bundle $\mathcal{V}$, there is a non-degenerate pairing

$$H^0(C, \mathcal{V}) \times H^1(C, \Omega_C \otimes \mathcal{V}^*) \longrightarrow H^1(C, \Omega_C) \cong k.$$

Theorem 8 (Riemann-Roch): With $g := \dim_k \Omega_C(C)$, we have for every vector bundle $\mathcal{V}$:

$$\chi(C, \mathcal{V}) = \deg(\det \mathcal{V}) + \dim(\mathcal{V})(1-g)$$

3.2 First proofs

By induction one can show for all $D \in \operatorname{Div}(C)$:

$$\chi(C, \mathcal{O}_C(D)) = \deg(D) + \chi(C, \mathcal{O}_C),$$

implying Thm 7a). Using 7b) and 7c) we have $\chi(C, \mathcal{O}_C) = 1-g$ giving Thm 8.
~~with~~ It remains to show 7b) and 7c).

Fact 3.2.4: Theorem 7b) and 7c) hold for $C = \mathbb{P}^1_k$.
Proof: By theorem 2, it is enough to show $\Omega_{\mathbb{P}^1_k} \cong \mathcal{O}(-2)$, by giving a universal derivation $d: \mathcal{O}_{\mathbb{P}^1_k} \to \mathcal{O}(-2)$.

3.3 The functor $f^!$

Let $f: C \to D$ be a finite separable morphism of conn. regular curves of finite type.

Fact 3.3.1: There is a sheaf of Dedekind differents $\mathfrak{D}_f \subseteq \mathcal{O}_C$:

$$\mathfrak{D}_f(f^{-1}U) = \mathfrak{D}_{\mathcal{O}_C(f^{-1}(U))/\mathcal{O}_D(U)} \quad \text{for } U \subseteq D \text{ affine open}$$

Prop 3.3.1 $f^!: \operatorname{Qc}(D) \to \operatorname{Qc}(C): M \longmapsto f^*M \otimes_{\mathcal{O}_C} \mathfrak{D}_f^{-1}$ is <u>right adjoint</u> to f_*:
That is:

$$\operatorname{Hom}_{\mathcal{O}_C}(N, f^*M \otimes_{\mathcal{O}_C} \mathfrak{D}_f^{-1}) \cong \operatorname{Hom}_{\mathcal{O}_D}(f_*N, M).$$

Prop 3.3.3: We have $f^!\Omega_D \cong \Omega_C$.
Prop 3.3.4: This iso behaves well with dlog, Tr_f and the norm.

3.4 Proof Serre Duality for curves

Prop 3.4.1: There is a finite separable morphism $f: C \to \mathbb{P}^1_k$

Proof theorem 7b) and 7c): Using this $f: C \to \mathbb{P}^1_k$, we get by fact 3.2.4

$$H^1(C, \mathcal{V})^* \cong H^1(\mathbb{P}^1_k, f_*\mathcal{V})^* \cong \operatorname{Hom}(f_*\mathcal{V}, \Omega_{\mathbb{P}^1_k}) \cong \operatorname{Hom}(\mathcal{V}, f^!\Omega_{\mathbb{P}^1_k}) \cong \operatorname{Hom}(\mathcal{V}, \Omega_C) = H^0(C, \Omega_C \otimes \mathcal{V}^*).$$

In particular, $H^1(C, \Omega_C) \cong k$. Also the square in 7(b) commutes.

Algebraic Geometry II – Overview of theorems and definitions

1. Cohomology of q.c. sheaves of modules

1.1 Recollection of basic definitions and results

Def 1.1.1: Preschemes and schemes (separated preschemes)

Prop 1.1.1: If X is a scheme and $U, V \subseteq X$ affine opens, then $U \cap V$ is affine open

Def 1.1.2: a) Pointed category, b) Additive category c) Abelian category

Ex 1.1.1 The category of sheaves of $\mathcal{R}$-modules is an Abelian category

Prop 1.1.2: If $f: X \to Y$ is q.c. and q.s. and $\mathcal{M} \in \underline{Qc}(X)$, then $f_*\mathcal{M} \in \underline{Qc}(Y)$

Prop 1.1.3: a) $\underline{Qc}(X)$ is closed under kernels, cokernels and finite sums.
b) If $\mathcal{M} \in \underline{Qc}(X)$ and $U \subseteq X$ open, then $\mathcal{M}|_U \in \underline{Qc}(U)$

Prop 1.1.4: Bijection

$$\begin{array}{ccc} \{\text{closed subpreschemes of } X\} & \xrightarrow{\simeq} & \{\text{q.c. ideals of } \mathcal{O}_X\} \\ (f: Y \to X) & \longmapsto & \ker(\mathcal{O}_X \xrightarrow{f^*} f_*\mathcal{O}_Y) \end{array}$$

Def 1.1.3: a) Locally finitely generated $\mathcal{O}_X$-module $\mathcal{M}$
b) Coherent $\mathcal{O}_X$-module $\mathcal{M}$ (on loc. Noetherian prescheme)

Cor 1.1.1: If $f: \mathcal{M} \to \mathcal{N}$ is a morphism of q.c. $\mathcal{O}_X$-modules, then $\ker(f)$ and $\operatorname{coker}(f)$ give the expected thing on affine open subsets $U \subseteq X$.

Cor 1.1.2: Taking local sections in $U \subseteq X$ affine open is an exact functor.

1.2. Čech cohomology

Def 1.2.1: For $\mathcal{U}: X = \bigcup_{i \in I} U_i$ open cover of X, $\mathcal{M}$ a q.c. $\mathcal{O}_X$-module, we have the Čech complex $\check{C}^*(\mathcal{U}, \mathcal{M})$

$$\check{C}^k(\mathcal{U}, \mathcal{M}) = \prod_{(i_0 \dots i_k) \in I^{k+1}} \mathcal{M}(U_{i_0 \dots i_k})$$

with $\check{d} = \sum_{j=0}^{k+1} (-1)^j d_j : \check{C}^k \to \check{C}^{k+1}$, $(d_j \varphi)_{i_0 \dots i_{k+1}} = \varphi_{i_0 \dots \hat{j} \dots i_{k+1}}|_{U_{i_0 \dots i_{k+1}}}$.

We have the antisymmetric subcomplex ${}^a\check{C}^*(\mathcal{U}, \mathcal{M})$.

Ex 1.2.1: a) The zeroth cohomology is $\check{H}^0(\mathcal{U}, \mathcal{M}) = {}^a\check{H}^0(\mathcal{U}, \mathcal{M}) = \mathcal{M}(X)$.
b) If $\mathcal{U} \ni : X = X$, we have trivial cohomology
c) $\check{C}^*$ distributes over direct sums and preserves short exact sequences in $\underline{Qc}(X)$.

Prop 1.2.1 For a s.e.s. $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ there is a long exact cohomology sequence:

$$\begin{array}{ccccccc} & \mathcal{M}'(X) & & \mathcal{M}(X) & & \mathcal{M}''(X) & \\ & \wr\wr & & \wr\wr & & \wr\wr & \\ 0 \to & \check{H}^0(\mathcal{U}, \mathcal{M}') & \to & \check{H}^0(\mathcal{U}, \mathcal{M}) & \to & \check{H}^0(\mathcal{U}, \mathcal{M}'') & \to \\ \to & \check{H}^1(\mathcal{U}, \mathcal{M}') & \to & \check{H}^1(\mathcal{U}, \mathcal{M}) & \to & \check{H}^1(\mathcal{U}, \mathcal{M}'') & \to \dots \end{array}$$

if $\mathcal{U}$ is an affine open cover of a scheme X (prop 1.1.1 and cor 1.1.2)

Def 1.2.2: Refinement of open cover and refinement maps

Lemma 1.2.1: a) Naturality of refinement maps
b) Common refinement of $\mathcal{U}$ and $\mathcal{V}$
c) All refinement maps are homotopic as chain maps

Cor 1.2.1 a) For $\mathcal{V}$ refinement of $\mathcal{U}$, there is canonical $\tau_{\mathcal{U}\mathcal{V}}: \check{H}^*(\mathcal{U}, \mathcal{M}) \to \check{H}^*(\mathcal{V}, \mathcal{M})$
b) It's an isomorphism if $\mathcal{U}$ also refinement of $\mathcal{V}$
c) If $U_{i_0} = X$, $\check{H}^k(\mathcal{U}, \mathcal{M}) = 0$ for $k > 0$.

Lemma 1.2.2: a) For continuous $f: X \to X$, $f^{-1}(\mathfrak{U}) := X = \bigcup f^{-1}(U_i)$

$$\check{C}^*(f^{-1}\mathfrak{U}, M) \cong \check{C}^*(\mathfrak{U}, f_* M)$$

b) If $im(f) \subseteq U_{i_0}$, $\check{H}^i(\mathfrak{U}, f_*\mathcal{F}) = 0$ for $i > 0$.

Prop 1.2.2 a) If X q.c. scheme, $\mathfrak{U}$ affine open cover, $\mathfrak{V}$ ~~subcover~~ affine refinement, M q.c. $\mathcal{O}_X$-module, then Tuv ~~is an~~ : $\check{H}^*(\mathfrak{U}, M) \to \check{H}^*(\mathfrak{V}, M)$ is an iso.

b) In the above situation, ${}^{a}\check{H}^*(\mathfrak{U}, M) \to \check{H}^*(\mathfrak{U}, M)$ is an iso.

c) If X is affine, then for $i > 0$

$$\check{H}^i(\mathfrak{U}, M) = {}^{a}\check{H}^i(\mathfrak{U}, M) = 0$$

Def 1.2.3: For q.c. scheme X, q.c. $\mathcal{O}_X$-module M, let $\mathfrak{U} :=$ {all open affines} and set

$$H^i(X; M) := \check{H}^i(\mathfrak{U}, M)$$

Theorem 1: a) Any open affine cover $\mathfrak{V}$ of a q.c. scheme X can be used to calculate $H^*(X; M)$ i.e.

$$H^*(X, M) \xrightarrow[\cong]{\mathrm{Tuv}} \check{H}^*(\mathfrak{V}, M) \xleftarrow{\cong} {}^{a}\check{H}^*(\mathfrak{V}, M)$$

are isomorphisms compatible with refinement of $\mathfrak{V}$.

b) When X is affine, $H^i(X, M) = 0$ for $i > 0$

c) There is a canonical isomorphism

$$H^0(X, M) \xrightarrow{\cong} M(X)$$

d) When $0 \to M' \to M \to M'' \to 0$ is a s.e.s. of q.c. $\mathcal{O}_X$-modules, one has a long exact cohomology sequence:

$$0 \to H^0(X, M') \to H^0(X, M) \to H^0(X, M'') \to H^1(X, M') \to H^1(X, M) \to H^1(X, M'') \to \cdots$$

Cor 1.2.2: Let X be any prescheme, and $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules.
If two of the three are q.c., so is the third, and for any affine open $U \subseteq X$ we have the exact sequence

$$0 \to M'(U) \to M(U) \to M''(U) \to 0.$$

Cor 1.2.3: Let X be a q.c. scheme.

a) Let $j: W \hookrightarrow X$ be the embedding of open affine $W \subseteq X$ and let $M \in Qc(W)$. Then for $p > 0$

$$H^p(X, j_* M) = 0$$

b) For $W = \bigcup_{i=1}^{n} W_i \subseteq X$, all W_i affine, and $M \in Qc(X)$, we have

$$H^p(X, \bigoplus_{i=1}^{n} (j_{W_i})_*(M|_{W_i})) = 0 \quad \text{for } p > 0$$

and the kernel of

$$M \longrightarrow \bigoplus_{i=1}^{n} (j_{W_i})_*(M|_{W_i})$$

$$m \in M(U) \longmapsto (m|_{U \cap W_i})_{i=1}^{n}$$

vanishes on W.

Lemma 1.2.3: Let X be a q.s. prescheme, $\mathfrak{U}$ a finite open cover of q.c. sets and $f \in \mathcal{O}_X(X)$. Then we have an iso for q.c. $M \in Qc(X)$

$$\check{C}(\mathfrak{U}, M)_f \xrightarrow{\cong} \check{C}(\mathfrak{U} \cap (X \setminus V(f)), M|_{X \setminus V(f)})$$

Prop 1.2.3 Let X be a q.c. scheme, M a q.c. $\mathcal{O}_X$-module, $f \in \mathcal{O}_X(X)$. Then

$$H^*(X, M)_f \xrightarrow{\cong} H^*(X \setminus V(f), M|_{X \setminus V(f)})$$

1.3 The affinity criterion of Serre

Prop 1.3.1 (Serre's affinity criterion) For a q.c. scheme X, t.f.a.e.:

a) X is affine

b) For all q.c. sheafs of $\mathcal{O}_X$-modules $\mathcal{M}$ and $p>0$, $H^p(X,\mathcal{M})=0$

c) For any q.c. sheaf of ideals $\mathcal{J}$ on X, $H^1(X,\mathcal{J})=0$

Prop 1.3.2: a) If Z is a q.c. closed subset of a prescheme X, it contains a closed point.

b) If $Z\subseteq X$ is a closed subset of a prescheme X then
$$\mathcal{J}(U) := \{f\in\mathcal{O}_X(U) \mid Z\cap U\subseteq V(f)\}$$
is a q.c. sheaf of ideals on X.

c) If $\mathcal{N}_1,\mathcal{N}_2\subseteq\mathcal{M}$ are q.c. subsheafes of the q.c. $\mathcal{O}_X$-module $\mathcal{M}$ then
$$(\mathcal{N}_1\cap\mathcal{N}_2)(U) := \mathcal{N}_1(U)\cap\mathcal{N}_2(U)$$
defines a q.c. sheaf of modules.

Proof: c) $\Rightarrow$ a) By induction and using prop 1.3.2, for every q.c. submodule $\mathcal{M}\subseteq\mathcal{O}_X^n$ we have $H^1(X,\mathcal{M})=0$, using $0\to\mathcal{M}\cap\mathcal{O}_X^{n-1}\to\mathcal{M}\to\mathcal{M}/_{\mathcal{M}\cap\mathcal{O}_X^{n-1}}\to 0$ and l.e.s.

For $R=\mathcal{O}_X(X)$ we get a map $p: X\to \operatorname{Spec} R=\operatorname{Spec}\mathcal{O}_X(X)$ corresp. to $\mathrm{id}: R\to R$.

Show: α) If $f\in R$ st. $X\setminus V(f)$ affine, then $p|_{\dots}: X\setminus V(f)=p^{-1}(R\setminus V(f))\xrightarrow{\cong}\operatorname{Spec}R\setminus V(f)$ is iso.

β) Such open affine $X\setminus V(f)$ cover X. (If not, take closed point z outside of their union (by 1.3.2) and let V be an affine open nbhd. Find sheaf of ideals $\mathcal{J}$ corresponding to $Y=\{z\}\cup X\setminus V=Y_2\sqcup Y_1$, and $\varphi\in\mathcal{O}_Y(Y)$ st. $\varphi|_{Y_1}=0$, $\varphi|_{Y_2}=1$. By $\mathcal{O}_X(X)\to(i_*\mathcal{O}_Y)(X)\to H^1(X,\mathcal{J})=0$, φ lifts to $f\in\mathcal{O}_X(X)=R$, so $Y_1\subseteq V(f)$, $z\notin V(f)$ so $z\in X\setminus V(f)=V\setminus V(f)$ is of the right form.)

γ) If $X=\bigcup_{i=1}^n X\setminus V(f_i)$ with f_i as in α), then $\operatorname{Spec}(R)=\bigcup_{i=1}^n\operatorname{Spec}R\setminus V(f_i)$ ($\mathcal{O}_X^n\xrightarrow{(f_1,\dots,f_n)}\mathcal{O}_X$ is epi since $\mathcal{O}_X\xrightarrow{f_i}\mathcal{O}_X$ is iso on $V_i=X\setminus V(f_i)$ and these cover X. Letting $\mathcal{M}$ be it's kernel, $\mathcal{M}(X)\to\mathcal{O}_X^n(X)\xrightarrow{(f_1,\dots,f_n)}\mathcal{O}_X(X)\to H^1(\mathcal{M})=0$ shows that the f_i generate R so $\bigcap_{i=1}^n V(f_i)=\emptyset$ in $\operatorname{Spec}R$.)

Done.

1.4 Cohomological dimension

Prop. 1.4.1: (Grothendieck) Let X be a scheme, $Z\subseteq X$ closed subset which is Noetherian. Then $H^p(X,\mathcal{M})=0$ when $\mathcal{M}\in\mathcal{Q}c(X)$ st. $\mathcal{M}|_{X\setminus Z}=0$ and $p>\dim(Z)$.

Proof: Induction on $d=\dim(Z)$. Write $Z=\bigcup_{i=1}^n Z_i$, $\eta_i\in Z_i$ generic point of irred. comp Z_i. By minimality of $\bigcup_{i=1}^n Z_i$, $\eta_j\notin Z_i$ if $j\neq i$, so there are affine open nbhds W_i of the η_i st. $W_i\cap Z_j=\emptyset$. Let $j_i: W_i\hookrightarrow X$ open immersion, $\mathcal{M}_i=(j_i)_*(\mathcal{M}|_{W_i})$, $\hat{\mathcal{M}}=\bigoplus_{i=1}^n\mathcal{M}_i$.

By cor 1.2.3, $H^p(X,\hat{\mathcal{M}})=0$ for $p>0$, so it is enough to show $\mathcal{M}\xrightarrow{\cong}\hat{\mathcal{M}}$. Also by cor 1.2.3 $\ker(\mathcal{M}\to\hat{\mathcal{M}})$ vanishes on $W=\bigcup_{i=1}^n W_i$, but also on $X\setminus Z$ as $\mathcal{M}|_{X\setminus Z}\cong\hat{\mathcal{M}}|_{X\setminus Z}=0$.

For $d=0$, $Z_i=\{\eta_i\}$ and $W\cup X\setminus Z=X$, so $\ker(\mathcal{M}\to\hat{\mathcal{M}})=0$. Also $\mathcal{M}\to\hat{\mathcal{M}}$ is epi on stalks, so is iso.

For $d>0$, let $\hat{Z}=Z\cap X\setminus W$, so $\mathcal{M}\to\hat{\mathcal{M}}$ is mono on $\cancel{X}\setminus\hat{Z}$. Assume for now it's also epi on $X\setminus\hat{Z}$. As $\eta_i\notin\hat{Z}\subseteq X\setminus W$, no irred. comp. Z_i of Z is contained in $\hat{Z}$ so $\dim(\hat{Z})<\dim(Z)=d$. By induction, $H^p(X,\mathcal{K})=H^p(X,\mathcal{C})=0$, $\mathcal{K}$ and $\mathcal{C}$ being the (co-)kernel of $\mathcal{M}\to\hat{\mathcal{M}}$, for $p>d-1$. Then $0\to\mathcal{K}\to\mathcal{M}\to\hat{\mathcal{M}}\to\mathcal{C}\to 0$ splits as $0\to\mathcal{K}\to\mathcal{M}\to\mathcal{B}\to 0$ and $0\to\mathcal{B}\to\hat{\mathcal{M}}\to\mathcal{C}\to 0$, giving $\to H^{p-1}(X,\mathcal{C})\to H^p(X,\mathcal{B})\to H^p(X,\hat{\mathcal{M}})\to$ and $\to H^p(X,\mathcal{K})\to H^p(X,\mathcal{M})\to H^p(X,\mathcal{B})\to$ showing consequentially $H^p(X,\mathcal{B})=0$ and $H^p(X,\mathcal{M})=0$ for $p>d$.

Remains to prove that $\mathcal{C}|_{X\setminus\hat{Z}}=0$, i.e. $\mathcal{M}_x\to\hat{\mathcal{M}}_x$ epi on $X\setminus\hat{Z}$. For $x\in X\setminus Z$, it's clear, so let $x\in Z\setminus\hat{Z}=Z\cap W$, say $x\in Z\cap W_i$, then $x\in Z_i\cap W_i$, so $(\mathcal{M}_i)_x\cong\mathcal{M}_x$, $(\mathcal{M}_j)_x=0$, so $\mathcal{M}_x\to\hat{\mathcal{M}}_x$ also epi. □

1.5 Higher direct images (cohomology of morphisms)

For a q.c. separated morphism $f: X \longrightarrow Y$ of prescheme define the p-th direct image of the q.c. $\mathcal{O}_X$-module M under f, written $R^p f_* M$, as the sheafification of

$$U \longmapsto H^p(f^{-1}(U), M|)$$

defined on the set of q.c. open subschemes $U \subseteq Y$ (so that $f^{-1}(U)$ is a q.c. scheme.)

Prop 1.5.1: a) $R^0 f_* M = f_* M$ b) The $\mathcal{O}_Y$-modules $R^p f_* M$ are quasi-coherent

c) For a ses $0 \to M' \to M \to M'' \to 0$, there is a les $\cdots \to R^p f_* M' \to R^p f_* M \to R^p f_* M'' \to R^{p+1} f_* M' \to R^{p+1} f_* M \to \cdots$

d) When U is affine, the canonical morphism

$$\boxed{(1) \qquad H^p(f^{-1}(U), M) \longrightarrow R^p f_* M(U)}$$

is an isomorphism.

Proof: a) Trivial. c) Theorem 1. b) & d) Wma Y is affine, $U = Y = \mathrm{Spec}(R)$. Then $U = Y \setminus V(\varphi)$ for $\varphi \in R$ form a topology base of Y and by prop 1.2.3 we have $H^p(f^{-1}U, M) = H^p(f^{-1}(Y \setminus V(\varphi)), M)$ $= H^p(X \setminus V(f^*\varphi), M) = H^p(X, M)_{f^*\varphi} = M_{f^*\varphi}$, where $M = H^p(X, M)$. Regarding the $\mathcal{O}_X(X)$-module M as an R-module, we see that $R^p f_* M$ is canonically isomorphic to the sheafification of $U = Y \setminus V(\varphi) \mapsto M_\varphi$ which is q.c., so we have b). Also d) follows as we know. □

There is also the "pull-back" morphism

$$\boxed{(2) \qquad H^p(Y, f_* M) \longrightarrow H^p(X, M)}$$

defined by taking affine open cover $\mathcal{U}$ of Y, letting $\mathcal{V}$ affine refinement of $f^{-1}\mathcal{U}$ and

$$H^p(Y, f_* M) \xrightarrow{\cong} \check{H}^p(\mathcal{U}, f_* M) \overset{\text{lem 1.2.2}}{\cong} \check{H}^p(f^{-1}\mathcal{U}, M) \xrightarrow{\text{refine}} \check{H}^p(\mathcal{V}, M) \xleftarrow{\cong} H^p(X, M).$$

The morphisms (1) and (2) in general fail to be isomorphisms but are part of the Leray Spectral sequence $E_2^{pq} = H^p(Y, R^q f_* M)$ converging to $H^{p+q}(X, M)$

Note $E_2^{p,0} = H^p(Y, f_* M)$ occurring in (2), $E_2^{0,q} = H^0(Y, R^p f_* M) = R^p f_* M(Y)$ occurring in (1).

More generally there is a Leray Spectral sequence $R^p g_* R^q f_* M \longrightarrow R^{p+q}(gf)_* M$.

1.6 Affine morphisms

Prop 1.6.1: For a (q.c. and q.s.) morphism $f: X \longrightarrow Y$ with X q.s., t.f.a.e:

a) For all $U \subseteq Y$ affine open, $f^{-1}(U)$ is affine open

b) It is possible to cover Y with affine open U st. $f^{-1}(U)$ affine.

c) The map f is separated (and q.c.) and for all q.c. $\mathcal{O}_X$-modules M, $p > 0$, $R^p f_* M = 0$.

Proof: b) ⇒ c) Thm 1 + prop 1.5.1 say $R^p f_* M(U) = H^p(f^{-1}(U), M) = 0$ on top. base of Y.

c) ⇒ a) Idea: Serre's criterion. Let $U \subseteq Y$ affine open, $M \in Qc(f^{-1}U)$. Then $j: f^{-1}U \hookrightarrow X$ is q.s. and q.c. (as X is q.s. and V is q.c.) so $\tilde{M} := j_* M \in Qc(X)$ by prop 1.1.2. Then $H^p(f^{-1}U, M) \cong H^p(f^{-1}U, \tilde{M}|_{f^{-1}U}) \overset{\text{prop 1.5.1}}{\cong} (R^p f_* M)(U) = 0$, so by Serre's criterion $f^{-1}(U)$ affine.

Def 1.6.1: a) Such $f: X \longrightarrow Y$ is called affine

b) $f: X \longrightarrow Y$ is finite if in addition $\mathcal{O}_X(f^{-1}U)$ is a f.g. $\mathcal{O}_Y(U)$-module for all affine $U \subseteq Y$ (i.e. $f_* \mathcal{O}_X$ is a locally finitely generated $\mathcal{O}_Y$-module.)

Cor 1.6.1: For affine morphism $f: X \longrightarrow Y$ and q.c. $\mathcal{O}_X$-module M, $R^p f_* M = 0$ for $p > 0$ and the map (2) in 1.5 is an isomorphism: $H^p(Y, f_* M) \cong H^p(X, M)$.

Proof: For all U, $f^{-1}(U)$ is affine so $H^p(f^{-1}(U), M)$ vanishes and thus does $R^p f_* M$. Also for open affine cover $\mathcal{U}$ of Y, $f^{-1}\mathcal{U}$ is already affine cover of X, so hence the iso.

Definition: Let A be an q.c. $\mathcal{O}_Y$-algebra. We define $\mathrm{Spec}\, A \longrightarrow Y$. For all $U \subseteq Y$ affine open, we have $\mathrm{Spec}\, A(U) \longrightarrow \mathrm{Spec}(\mathcal{O}_Y(U)) \cong U$ behaving well under taking subsets $V \subseteq U$ so we can glue them together to $\mathrm{Spec}\, A \longrightarrow Y$.

Alternatively it can be described explicitly as a space with points $(y, \mathfrak{p})$ where $\mathfrak{p} \in \mathrm{Spec}\, A_y$ st. $\mathfrak{p} \cap \mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$ and topology base $\Omega(U, \lambda) = \{(y, \mathfrak{p}) \mid y \in U, \lambda_y \notin \mathfrak{p}\}$ for $U \subseteq Y$ affine and $\lambda \in A(U)$. The local ring at $(y, \mathfrak{p})$ is $(A_Y)_\mathfrak{p}$ and sections ~~are~~ of the structure sheaf are tuples $(f_{[y,\mathfrak{p}]})_\Omega$ st. locally in small open nbhds $\Omega(V, \lambda)$ (with $V \subseteq Y$ affine, $\lambda \in A(V)$) there is $\varphi \in A(V)_\lambda$ st. each $f_{[y,\mathfrak{p}]}$ is the image of φ under $A(V)_\lambda \longrightarrow (A_Y)_\lambda \longrightarrow (A_Y)_\mathfrak{p}$.

One has a bijection

(2) $$\mathrm{Hom}_{Y\text{-presc}}(X, \underline{\mathrm{Spec}}\, \mathcal{A}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_Y\text{-alg}}(\mathcal{A}, \xi_* \mathcal{O}_X)$$

for any morphism $\xi: X \to Y$ of preschemes by gluing the bijections

$$\mathrm{Hom}(\xi^{-1}U, \mathrm{Spec}(\mathcal{A}(U))) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{A}(U), \mathcal{O}_X(\xi^{-1}(U)))$$

Prop 1.6.2 a) The morphism $\underline{\mathrm{Spec}}\, \mathcal{A} \longrightarrow Y$ is affine

b) (2) is a bijection

c) $\xi: X \to Y$ is affine iff $\mathcal{A} := \xi_* \mathcal{O}_X$ is q.c. $\mathcal{O}_Y$-module and $X \longrightarrow \underline{\mathrm{Spec}}\, \xi_* \mathcal{O}_X$ corresponding to $\mathrm{id}_{\mathcal{A}}$ under (2) is an iso.

Cor 1.6.2: For any $f: \tilde{Y} \to Y$, $\underline{\mathrm{Spec}}(f^* \mathcal{A}) \cong \underline{\mathrm{Spec}}\, \mathcal{A} \times_Y \tilde{Y}$, (by b))

1.7 The relation between H^1 and torseurs

Def 1.7.1: For a top. space X and sheaf of groups $\mathcal{G}$, a $\mathcal{G}$-torseur is a sheaf of sets $\mathcal{X}$ on X with a morphism $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$ s.t. $1 \cdot \xi = \xi$, $g(h\xi) = (gh)\xi$, and s.t. the action of $\mathcal{G}_x$ on $\mathcal{X}_x \neq \emptyset$ is simply transitive for all $x \in X$. The torseur is called trivial if $\mathcal{X}(X) \neq \emptyset$, in which case $\mathcal{X} \cong \mathcal{G}$ (as $\mathcal{G}$-torseurs)

Ex 1.7.2 a) For $\mathcal{L}$ a line bundle on the l.r.s. X, let for all $\ell \in \mathcal{L}(U)$

$$V(\ell) = \{x \in U \mid \text{image of } \ell \text{ in } \mathcal{L}_x \text{ is in } \mathfrak{m}_x \mathcal{L}_x\}$$

We get an $\mathcal{O}_X^*$-torseur by setting

$$\mathcal{L}^*(U) = \{\ell \in \mathcal{L}(U) \mid V(\ell) = \emptyset\} = \{\ell \mid \ell \text{ free generator of } \mathcal{L}|_U\},$$

which is indeed invariant under left-multiplication by $\mathcal{O}_X^*$.

One can show that each $\mathcal{O}_X^*$-torseur is of this form, so

$$\left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of line bundles on } X\end{array}\right\} \xleftrightarrow{\cong} \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of } \mathcal{O}_X^*\text{-torseurs}\end{array}\right\}$$

b) Similarly, we have for

$$(GL_n \mathcal{O}_X)(U) = \{g \in \mathrm{Mat}_{n,n}(\mathcal{O}_X(U)) \mid V(\det g) = \emptyset\}$$
$$= \{g \in \mathrm{Mat}_{n,n}(\mathcal{O}_X(U)) \mid g \text{ has inverse matrix}\}$$

the bijection

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ n\text{-dim vector bundles on } X\end{array}\right\} \xleftrightarrow{\cong} \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of } GL_n(\mathcal{O}_X)\text{-torseurs}\end{array}\right\}$$

Now let $\mathcal{G}$ be a sheaf of abelian groups, $\mathcal{U}: X = \bigcup_{i \in I} U_i$ an open cover and $\mathcal{X}$ a $\mathcal{G}$-torseur trivial on $\mathcal{U}$, (i.e. $\mathcal{X}|_{U_i}$ trivial).

Let $\xi_i \in \mathcal{X}(U_i)$ be elements and define $\psi = (\psi_{ij}) \in \check{C}^1(\mathcal{U}, \mathcal{G})$ by

$$\psi_{ij} = \xi_i|_{U_{ij}} - \xi_j|_{U_{ij}} \in \mathcal{G}(U_{ij})$$

Prop 1.7.1 a) $d(\psi) = 0$, and the cohomology class $[\psi] \in \check{H}^1(\mathcal{U}, \mathcal{G})$ is indep. of ξ_i.

b) There is a bijection between $\mathcal{G}$-torseurs trivial on $\mathcal{U}$ and $\check{H}^1(\mathcal{U}, \mathcal{G})$.

Cor 1.7.1: If X affine, $\mathcal{M}$ a q.c. $\mathcal{O}_X$-module, then all $\mathcal{M}$-torseurs are trivial.

Def 1.7.2: If X is an S-prescheme, we have a homomorphism of sheaves of abelian groups $\mathcal{O}_X^* \xrightarrow{\mathrm{dlog}} \Omega_{X/S}$, $\mathrm{dlog}(f) = d_{X/S} f / f$, defining

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \Omega_{X/S}).$$

For a line bundle $\mathcal{L}$, the $\mathcal{O}_X^*$ torseur ~~bundle~~ $\mathcal{L}^*$ corresponds to an element in $H^1(X, \mathcal{O}_X^*)$, which is mapped to an element $c(\mathcal{L}) \in H^1(X, \Omega_{X/S})$, called the first Chern class of $\mathcal{L}$

Prop 1.7.2 For vector-bundles $\mathcal{V}, \mathcal{W}$ over the l.r.s. X,

$$\mathrm{Ext}(\mathcal{W}, \mathcal{V}) \cong \{\mathcal{H}om(\mathcal{W}, \mathcal{V})\text{-torseurs on } X\}$$

2. Cohomology of Projective Spaces

2.1 Regular sequences and the Koszul complex

Def 2.1.0 a) For a chain complex (C^*, d_C^*), its shift $C^*[p]$ is defined as $(C^*[p])^q = C^{p+q}$, $d_{C^*[p]}^q = (-1)^p d_{C^*}^{p+q}$. Clearly $H^q(C^*[p]) = H^{p+q}(C^*)$.

b) For $\varphi : C^* \to \hat{C}^*$, define its cone $\mathrm{Cone}(\varphi)$ by
$$\mathrm{Cone}(\varphi)^p = \hat{C}^p \oplus C^{p+1}$$
$$d = \begin{pmatrix} d_{\hat{C}} & \varphi \\ 0 & -d_C \end{pmatrix}$$
The ses $0 \to \hat{C}^* \to \mathrm{Cone}(\varphi) \to C[1]^* \to 0$ gives a les.
$$\dots \to H^p(C) \xrightarrow{\varphi} H^p(\hat{C}) \to H^p(\mathrm{Cone}(\varphi)) \to H^{p+1}(C) \xrightarrow{\varphi} H^{p+1}(\hat{C}) \to \dots$$
so φ induces isomorphisms in cohomology iff the chomology of $\mathrm{Cone}(\varphi)$ vanishes.

Def 2.1.1: Let R be a ring, M an R-module. A sequence $(x_0, \dots, x_n) \in R^{n+1}$ is called M-regular if for $0 \le i \le n$ the map
$$M/(x_0M + \dots + x_{i-1}M) \xrightarrow{\cdot x_i} M/(x_0M + \dots + x_{i-1}M)$$
is injective. If $M = R$, we say $\underline{x} = (x_0, \dots, x_n)$ is regular.

Ex 2.1.1 For S a ring, $R = S[x_0, \dots, x_n]$, the sequence $(x_0, \dots, x_n)$ is R-regular.

Def 2.1.2: For a sequence $\underline{x} = (x_0, \dots, x_n) \in R^{n+1}$ and R-module M, define the Koszul-complex $C^*(\underline{x}, M)$ by
$$C^m(\underline{x}, M) = \left\{ \begin{array}{l} f : \underline{n}^m \to M \text{ satisfying} \\ \text{a) } f(i_1, \dots, i_m) = 0 \text{ if } i_k = i_\ell \\ \text{b) } f(i_{\sigma(1)}, \dots, i_{\sigma(m)}) = \mathrm{sign}(\sigma) f(i_1, \dots, i_m) \end{array} \right\}$$
with differential $d = \sum_{j=0}^{m} (-1)^j d_j : C^m(\underline{x}, M) \to C^{m+1}(\underline{x}, M)$

where
$$(d_j f)(i_1, \dots, i_{m+1}) = x_{i_{j+1}} f(i_1, \dots, \hat{i}_{j+1}, \dots, i_{m+1})$$
Define $H^*(\underline{x}, M) = H^*(C^*(\underline{x}, M))$. Note that $C^k(\underline{x}, M) = 0$ for $k < 0$, $k > n+1$ and
$$H^0(\underline{x}, M) = \bigcap_{i=0}^{n} \ker(M \xrightarrow{\cdot x_i} M)$$
$$H^{n+1}(\underline{x}, M) \cong M/(x_0M + \dots + x_nM)$$
Note that a ses of R-modules gives a ses of Koszul-complexes.

Remark 2.1.3: There is a canonical isomorphism between $C^*((x_0, \dots, x_n), M)$ and $\mathrm{Cone}(C^*((x_0, \dots, x_{n-1}), M) \xrightarrow{\cdot x_n} C^*((x_0, \dots, x_{n-1}), M))[-1]$ sending f to $(f(n, -, \dots, -)|_{\underline{(n-1)}^{m-1}}, f|_{\underline{(n-1)}^m})$.

Fact 2.1.1 a) $(x_0, \dots, x_n)$ is M-regular iff $(x_0, \dots, x_{i-1})$ M-regular, $(x_i, \dots, x_n)$ $M/(x_0M + \dots + x_{i-1}M)$ regular.

b) $(x_0, \dots, x_n)$ is M regular iff $H^j((x_0, \dots, x_i), M) = 0$ for all $0 \le i \le n$, $j \ne i+1$

c) For ses $0 \to M' \to M \to M'' \to 0$, $\underline{x}$ is M-regular if its M'- and M''-regular

d) If $(x_0, \dots, x_n)$ is M-regular, then $(x_0^{k_0}, \dots, x_n^{k_n})$ is Mregular.

Def 2.1.3: For f.g. free R-modules M, N, a non-degenerate pairing is an R-bilinear map $M \times N \to R$, st. $M \to \mathrm{Hom}_R(N, R)$ is an iso (equivalently $N \to \mathrm{Hom}_R(M, R)$ is an iso.)

Cor 2.1.1: Let R be a ring, $S := R[X_0, \dots, X_n]$, $\underline{X}^\ell := (X_0^\ell, \dots, X_n^\ell)$.
Then $H^k(C^*(\underline{X}^\ell, S)) = 0$ for $k \neq n+1$, and $H^{n+1}(C^*(\underline{X}^\ell, S))$ is a free R-module.
The Koszul-complex becomes a complex of graded S-modules via
$$C^m(\underline{X}^\ell, S)_k := \{ f \in C^m(\underline{X}^\ell, S) \mid f(i_1, \dots, i_m) \in S_{k+\ell m} \}$$
and
(a) $H^{n+1}(\underline{X}^\ell, S)_k = 0$ for $k > -n-1$
(b) $H^{n+1}(\underline{X}^\ell, S)_{-n-1} \cong R$ sending $[f: \underline{n}^{n+1} \to S]$ to the coefficient of $X_0^{\ell-1} \cdots X_n^{\ell-1}$ in $f(0, 1, \dots, n)$.

(c) For $0 < k < \ell$, we have a non-degenerate pairing
$$S_k \times H^{n+1}(\underline{X}^\ell, S)_{-n-1-k} \xrightarrow{\text{multiplic.}} H^{n+1}(\underline{X}^\ell, S)_{-n-1} \xrightarrow[\cong]{(b)} R$$
Lemma 2.1.1: For Noetherian local ring R, f.g. R-module M, $\underline{x} = (x_0, \dots, x_n) \in \mathfrak{m}^{n+1} \subseteq R^{n+1}$,
$\underline{x}$ is M-regular $\iff$ $H^i(\underline{x}, M) = 0$ for $i \neq n+1$.

2.2 The scheme $\mathbb{P}^n_A$ and its cohomology

For an $\mathbb{N}$-graded ring $R_\cdot$, we have the prescheme
$$\mathrm{Proj}(R_\cdot) = \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \text{ homogeneous}, R_+ \not\subseteq \mathfrak{p} \}$$
with closed sets $V(I) := \{ \mathfrak{p} \mid I \subseteq \mathfrak{p} \}$ for $I \subseteq R$ homogeneous. We have sheaf of graded rings $\mathcal{O}(-)$ on $\mathrm{Proj}(R_\cdot)$, where $\mathcal{O}(k)$ is sheafification of
$$U = \mathrm{Proj}(R_\cdot) \setminus V(f) \longmapsto (R_f)_k \qquad (f \in R_\cdot \text{ homogeneous})$$
In particular, $\mathcal{O}(0)$ is a sheaf of rings, the structure sheaf of $\mathrm{Proj}(R_\cdot)$
Prop 2.2.1: a) $\mathrm{Proj}(R_\cdot)$ is a scheme with affine topology base the open sets $\mathrm{Proj}(R_\cdot) \setminus V(f)$ for $f \in R_k$ homogeneous, $k > 0$.
b) Every finite subset of $\mathrm{Proj}(R_\cdot)$ is contained in an affine open subset.
c) The sheaf of modules $\mathcal{O}(k)$ is q.c. and
$$\mathrm{Proj}(R_\cdot) \setminus V(f) \xrightarrow{\cong} \mathrm{Spec}((R_f)_0)$$
$$\mathcal{O}(0)|_{\mathrm{Proj}(R_\cdot)\setminus V(f)} \xrightarrow{\cong} \mathcal{O}_{\mathrm{Spec}((R_f)_0)} = \widetilde{(R_f)_0}$$
$$\mathcal{O}(k)|_{\mathrm{Proj}(R_\cdot)\setminus V(f)} \xrightarrow{\cong} \widetilde{(R_f)_k}$$
d) When $R_\cdot$ is generated by R_1 as an R_0-algebra, then the $\mathcal{O}(k)$ are line bundles and we have an isomorphism
$$\mathcal{O}(k) \otimes_{\mathcal{O}(0)} \mathcal{O}(\ell) \xrightarrow{\cong} \mathcal{O}(k+\ell) \qquad f \otimes g \mapsto fg$$
e) We have a morphism $\pi: \mathrm{Proj}(R_\cdot) \to \mathrm{Spec}(R_0): \mathfrak{p} \mapsto \mathfrak{p} \cap R_0$
$$\mathcal{O}_{\mathrm{Spec}(R_0)} \to \mathcal{O}_{\mathrm{Proj}(R_\cdot)}: \frac{r}{f^k} \mapsto \frac{r}{f^k}.$$
If $R_\cdot/R_0$ is of finite type, then π is of f.t. and
$$\pi^{-1}(\mathrm{Spec}(R_0) \setminus V(f)) = \mathrm{Proj}(R_\cdot) \setminus V(f)$$
f) If $R_\cdot$ is Noetherian, $\mathrm{Proj}(R_\cdot)$ is Noetherian, and the $\mathcal{O}(k)$ are coherent.
Remark: For a graded $R_\cdot$-module M we have the $\mathcal{O}_{\mathrm{Proj}(R_\cdot)}$-module $\widehat{M}_\cdot$ as the sheafification of
$$U = \mathrm{Proj}(R_\cdot) \setminus V(f) \longmapsto \widetilde{(M_f)_0}$$
So $\widehat{M}_\cdot$ is q.c.
For $M = R[k]$, $\widehat{M} = \mathcal{O}(k)$.

Remark 2.2.3: A graded ring morphism $\varphi : R_\cdot \longrightarrow S_\cdot$ with $S_+ = \sqrt{S\cdot\varphi(R_+)}$ gives a morphism

$$\begin{aligned} \mathrm{Proj}(S_\cdot) &\longrightarrow \mathrm{Proj}(R_\cdot) \\ \mathfrak{p} &\longmapsto \varphi^{-1}(\mathfrak{p}) \end{aligned}$$

together with

$$(R_{\varphi^{-1}(\mathfrak{p})})_0 \xrightarrow{\varphi} (S_{\mathfrak{p}})_0$$

In particular for $A[X_0,\dots,X_n] \twoheadrightarrow A[X_0,\dots,X_m]$ $(m \le n)$, we get a ~~projec~~ closed immersion $\mathbb{P}_A^m \hookrightarrow \mathbb{P}_A^n$.

Remark: For a graded $R_\cdot$-module $M_\cdot$, $\widetilde{M_\cdot[k]} \cong \widetilde{M_\cdot} \otimes \mathcal{O}(k)$

In particular $\mathcal{O}(k) \otimes \mathcal{O}(\ell) \xrightarrow{\cong} \mathcal{O}(k+\ell)$ is iso.

Theorem 2: Let A be a ring, $R := A[X_0,\dots,X_n]$, $X := \mathbb{P}_A^n = \mathrm{Proj}(R_\cdot)$

a) For $k \ge 0$

$$\begin{aligned} R_k &\xrightarrow{\cong} \mathcal{O}(k)(X) \\ r &\longmapsto (\text{image of } r \text{ in } (R_{\mathfrak{p}})_k)_{\mathfrak{p}\in X} \end{aligned}$$

b) For $0 < p < n$, $H^p(X,\mathcal{O}(k)) = 0$ for all k.

c) For $p = n \ge 0$ and $k > -n-1$ we have $H^n(X,\mathcal{O}(k)) = 0$

d) There is an isomorphism $H^n(X,\mathcal{O}(-n-1)) \cong A$ defined by

$$\check{C}^n(\mathcal{U},\mathcal{O}(-n-1)) \cong R[(X_0\cdots X_n)^{-1}]_{-n-1} \longrightarrow A$$
$$\sum_{\alpha\in\mathbb{Z}^n} f_\alpha X_0^{\alpha_0}\cdots X_n^{\alpha_n} \longmapsto f_{(-1,-1,\dots,-1)}$$

e) For all $k \in \mathbb{Z}$, the groups $H^0(\mathcal{O}(k))$ and $H^n(X,\mathcal{O}(k))$ are f.g. free A-modules, and there is a non-degenerate pairing

$$H^0(X,\mathcal{O}(k)) \times H^n(X,\mathcal{O}(-n-1-k)) \longrightarrow H^n(X,\mathcal{O}(-n-1)) \overset{(d)}{\cong} A$$

Proof: Compare with Koszul complex $C^*((X_0^j,\dots,X_n^j), R)$ and use fact 2.1.1 and cor 2.1.1

Lemma 2.2.1: An element $g \in GL_{n+1}(A)$ acts on $H^n(\mathbb{P}_A^n, \mathcal{O}(-n-1))$ by multiplication by $\det(g)^{-1}$

Proof: Reduce to case of A being domain and then to alg. closed field.

Def 2.2.1: A line bundle $\mathcal{L}$ on a quasi-compact prescheme X is called ample if it satisfies the following two equivalent conditions:

(a) For every locally f.g. q.c. $\mathcal{O}_X$-module $\mathcal{M}$, and sufficiently ~~high~~ large m, there is an epimorphism

$$((\mathcal{L}^{-1})^{\otimes m})^{\oplus n} \longrightarrow \mathcal{M} \qquad (\text{some } n)$$

(b) For all $\mathcal{M}$ as above, $\mathcal{L}^{\otimes k} \otimes \mathcal{M}$ is generated by its global sections for sufficiently large k.

Remark 2.2.4: Isomorphism classes of line bundles get an Abelian group structure with tensor product as addition and $\mathcal{L}^{-1} := \mathcal{H}om(\mathcal{L},\mathcal{O}_X)$.

On $\mathbb{P}_A^n$, we have

$$\mathcal{O}(k) \cong \mathcal{H}om(\mathcal{O}(\ell),\mathcal{O}(\ell+k))$$

In general, $\mathcal{L} \otimes \mathcal{H}om(\mathcal{L},\mathcal{M}) \cong \mathcal{M}$.

Note that $\mathcal{O}_X$ is ample if X is affine.

<u>Definition 2.2.2</u>: For a line bundle $\mathcal{L}$ on a LRS X, $U \subseteq X$ open, $\lambda \in \mathcal{L}(U)$, put

$$V(\lambda) := \{x \in U \mid \text{the image of } \lambda \text{ in } \mathcal{L}_x \text{ is in } m_x \mathcal{L}_x\}$$

This is a closed subset of U. We have $V(\lambda \otimes \mu) = V(\lambda) \cup V(\mu)$.

<u>Lemma 2.2.2</u>: Let X be q.c. prescheme, $\mathcal{L}$ a line bundle with $\lambda \in \mathcal{L}(X)$, $\mathcal{M}$ q.c.

(a) If $m \in \mathcal{M}(X)$ has $m|_{X \setminus V(\lambda)} = 0$, then $\lambda^{\otimes k} \otimes m = 0 \in (\mathcal{L}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M})(X)$ for some k.

(b) If X is also q.s. and $m \in \mathcal{M}(X \setminus V(\lambda))$, then there is $k \in \mathbb{N}$ s.t.

$\lambda^{\otimes k} \otimes m \in (\mathcal{L}^{\otimes k} \otimes \mathcal{M})(X \setminus V(\lambda))$ extends to a global section on X.

<u>Proof</u>: When $\mathcal{L} = \mathcal{O}_X$, (a) ~~gives~~ is injectivity and (b) is surjectivity of

$$\mathcal{M}(X)_\lambda \longrightarrow \mathcal{M}(X \setminus V(\lambda)),$$

which holds by q.c.-ness of $\mathcal{M}$.

For (a), cover X by fin. many U_i on which $\mathcal{L}$ is trivial, apply previous case and take maximum of all k obtained this way.

For (b), similarly as above we get ℓ s.t. $\lambda^{\otimes \ell} \otimes m$ extends to U_i. As $U_i \cap U_j$ is q.c. by q.s.-ness of X, we can apply (a) to see that all extensions agree after multiplying by $\lambda^{\otimes k}$ for some k. By the sheaf axiom we get a global extension of $\lambda^{\otimes(k+\ell)} \otimes m$ in $\mathcal{L}^{\otimes(k+\ell)} \otimes \mathcal{M}$.

<u>Prop 2.2.2</u>: Let X be q.c. and q.s., $\mathcal{L}$ line bundle and $s_1, \dots, s_n \in \mathcal{L}(X)$ st. $U_i := X \setminus V(s_i)$ cover X, where U_i is <u>affine</u>. Then $\mathcal{L}$ is ample.

<u>Proof</u>: Let $\mathcal{M}$ be a locally f.g. q.c. $\mathcal{O}_X$-module. Then $M_i := \mathcal{M}(U_i)$ is f.g. over ~~$\mathcal{O}_X$~~ $\mathcal{O}_X(U_i)$, say by $m_{i,j}$ for $j = 1, \dots, n_i$. By lemma 2.2.2 b) we can choose k_0 st. all $s_i^{\otimes k_0} \otimes m_{ij}$ extend to global sections of $\mathcal{L}^{k_0} \otimes \mathcal{M}$ on X. Thus $\mathcal{L}^{\otimes k} \otimes \mathcal{M}$ generated by global sections for each $k > k_0$, thus $\mathcal{L}$ ample.

<u>Cor 2.2.1</u>: a) The line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n_A$ is ample. $(U_i := \mathbb{P}^n_A \setminus V(X_i))$

b) If X is quasi-affine (open subscheme of affine) then $\mathcal{O}_X$ is ample.

For a q.c. module $\mathcal{M}$ on $\mathbb{P}^n_A$, define the <u>Serre twists</u> $\mathcal{M}(k) := \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{P}^n_A}} \mathcal{O}(k)$

<u>Theorem 3</u>: Let A be a Noetherian ring, $X = \mathbb{P}^n_A$, $\mathcal{M}$ a ~~q.c.~~ coherent $\mathcal{O}_X$-module. Then the groups $H^p(X, \mathcal{M})$ are f.g. A-modules and vanish for $p > n$.

For $p > 0$, we have $H^p(X, \mathcal{M}(k)) = 0$ for sufficiently large k.

<u>Proof</u>: Clear for $p > n$. Proceed by downward induction, so case $p+1$ is known. By ampleness of $\mathcal{O}(1)$, there is epi $\mathcal{O}(-k)^{\ell} \to \mathcal{M}$ for k large enough, so we get s.e.s. $0 \to \mathcal{K} \to \mathcal{O}(-k)^{\oplus \ell} \to \mathcal{M} \to 0$ and thus l.e.s.

$$H^p(X, \mathcal{O}(-k))^{\oplus \ell} \cong H^p(X, \mathcal{O}(-k)^{\oplus \ell}) \longrightarrow H^p(X, \mathcal{M}) \longrightarrow H^{p+1}(X, \mathcal{K})$$

As A Noetherian, $\mathcal{K}$ is coherent so by ind. ass. $H^{p+1}(X, \mathcal{K})$ is f.g. over A. The same holds for the left side by theorem 2 (b) and (c). So also for $H^p(X, \mathcal{M})$. Similarly, we have (using ~~exact~~ flatness of $\mathcal{O}(m)$, this being a line bundle)

$$H^p(X, \mathcal{O}(m-k))^{\oplus \ell} \longrightarrow H^p(X, \mathcal{M}(m)) \longrightarrow H^{p+1}(X, \mathcal{K}(m))$$

By theorem 2 (b), (c), the left term vanishes for $p > 0$, $m > k-n$. The right term also vanishes for sufficiently large m and thus so does the middle term.

<u>Prop 2.2.3</u>: Let R be a Noetherian ring, X a closed subprescheme of $\mathbb{P}^n_R$. For sufficiently large k, the map

$$H^0(\mathbb{P}^n_R, \mathcal{O}(k)) = R[X_0, \dots, X_n]_k \longrightarrow H^0(X, \mathcal{O}(k)),$$

given by restriction, is surjective.

<u>Proof</u>: Let $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}^n_R}$ be the sheaf of ideals defining $X \subseteq \mathbb{P}^n_R =: Y$. We have a ses

$0 \to \mathcal{J}(k) \to \mathcal{O}_Y(k) \to i_*\mathcal{O}_X(k) \to 0$ (with $i : X \hookrightarrow Y$) giving us

$$H^0(Y, \mathcal{O}_Y(k)) \to H^0(X, \mathcal{O}_X(k)) \to H^1(Y, \mathcal{J}(k))$$
$$\cong H^0(Y, i_*\mathcal{O}_X(k))$$

and by Theorem 3, the right term vanishes for k large enough.

Prop 2.2.4 (Serre): Let A be Noetherian ring, $R = A[X_0, \dots, X_n]$, M graded R-module, M f.g., $\mathcal{M} = \tilde{M}$ on $\mathbb{P}^n_A$, so $\mathcal{M}$ is coherent.

a) When k sufficiently large, $M_k \to H^0(\mathbb{P}^n_A, \mathcal{M}(k)) \cong (\widetilde{M(k)})(\mathbb{P}^n_A)$ is an iso

b) For $A = k$ a field, define $\chi(\mathbb{P}^n_k, \mathcal{M}(k)) := \sum_{p=0}^{n} (-1)^p \dim_k (H^p(X, \mathcal{M}(k)))$

Then $\chi(X, \mathcal{M}(k)) = P_M(k)$, the Hilbert polynomial.

Proof: a) It's kernel is $N = \bigcup_{\ell=0}^{\infty} \{ m \in M \mid x_i^\ell m = 0 \text{ for all } i \} \subseteq M$, which is f.g. by Noetherianness of R, and thus vanishes for high k.

Surjectivity can be seen by taking a surjection $\bigoplus_{i=1}^{N} R[d_i] \to M$ giving a ses. $0 \to K \to \bigoplus_{i=1}^{N} \mathcal{O}_{\mathbb{P}^n_A}(d_i) \to \mathcal{M} \to 0$. As $H^1(\mathbb{P}^n_A, \mathcal{K}(k)) = 0$ for k sufficiently large, every element of $H^0(\mathbb{P}^n_A, \mathcal{M}(k))$ comes from $H^0(\mathbb{P}^n_A, \bigoplus_{i=1}^{N} \mathcal{O}_{\mathbb{P}^n_A}(d_i))$ $= \bigoplus_{i=1}^{N} (\mathcal{O}_{\mathbb{P}^n_A}(d_i + k))(\mathbb{P}^n_A) = \bigoplus_{i=1}^{N} R_{d_i + k}$, and thus from an element of M_k.

b) We know by theorem 3 that for large k $\chi(X, \mathcal{M}(k)) = \dim_k (H^0(X, \mathcal{M}(k)))$ $= \dim_k \tilde{M}_k = P_M(k)$, so it suffices to see $\chi(X, \mathcal{M}(k))$ is a polynomial in k.

Do this by induction on n. We have $0 \to K \to M \xrightarrow{\cdot X_n} M(1) \to Q \to 0$, giving us $\chi(M(\ell+1)) - \chi(M(\ell)) = \chi(Q(\ell)) - \chi(K(\ell))$. But K and Q are anihillated by X_n and thus come from $\mathbb{P}^{n-1}_A$, so ind. hypothesis applies, and $\chi(Q(\ell))$, $\chi(K(\ell))$ are polynomials. Thus so is $\chi(M(\ell))$.

Remark: (Line bundles and Weyl divisors on locally factorial Noeth. schemes)

Lemma: If all objects are f.g. graded k-vector spaces, then

a) $\sum_{i=-\infty}^{\infty} (-1)^i \dim_k H^i(C^\bullet) = \sum_{i=-\infty}^{\infty} (-1)^i \dim_k (C^i)$

b) For a l.e.s. $\dots \to C_{i-1} \to A_i \to B_i \to C_i \to A_{i+1}$, we have

$$\sum_{i=-\infty}^{\infty} (-1)^i \dim B^i = \sum_{i=-\infty}^{\infty} (-1)^i (\dim A_i + \dim C_i)$$

2.3 Projective morphisms

Let $\mathcal{R}_\bullet = \bigoplus_{i=0}^{\infty} \mathcal{R}_i$ be a graded q.c. $\mathcal{O}_X$-algebra over a prescheme X.

We define the prescheme $\underline{\mathrm{Proj}}(\mathcal{R}_\bullet)$ over X, essentially given by gluing $\mathrm{Proj}(\mathcal{R}(U_i)_\bullet) \to \mathrm{Spec}(\mathcal{R}(U_i)_0) \to \mathrm{Spec}(\mathcal{O}_X(U_i)) = U_i$ over all open affine $U_i \subseteq X$.

Explicitly, as a set

$$P := \underline{\mathrm{Proj}}(\mathcal{R}_\bullet) = \{ (x, \mathfrak{p}) \mid x \in X, \mathfrak{p} \in \mathrm{Spec}(\mathcal{R}_x), \text{ s.t. } \dots \}$$

where $\mathfrak{p} \subseteq \mathcal{R}_x$ is a homogeneous prime ideal not containing the augmentation ideal, with preimage $m_{X,x}$ under $\mathcal{O}_{X,x} \to (\mathcal{R}_0)_x \to \mathcal{R}_x$.

For a topology base, we take for $U \subseteq X$, $f \in \mathcal{R}_m(U)$

$$\Omega(U, f) = \{ (x, \mathfrak{p}) \mid x \in U, \text{ image of } f \text{ in } \mathcal{R}_x \text{ is not in } \mathfrak{p} \},$$

which satisfies $P = \Omega(X, 1)$, $\Omega(U, f) \cap \Omega(V, g) = \Omega(U \cap V, f|_{U \cap V} \cdot g|_{U \cap V})$

As structure sheaf, take sheafification of $\Omega(U, f) \mapsto (\mathcal{R}_\bullet(U)_f)_0 := (\bigoplus_{i=0}^{\infty} \mathcal{R}_i(U)_f)_0$

Explicitly, this has stalks $\mathcal{O}_{P,(x,\mathfrak{p})} = [(\mathcal{R}_x)_{\mathfrak{p}}]_0$ and thus

$$\mathcal{O}_P(W) = \{ (s_{(x,\mathfrak{p})}) \in \prod_{(x,\mathfrak{p})\in W} ((\mathcal{R}_x)_{\mathfrak{p}})_0 \mid \text{every } z\in W \text{ has open nbhd } V = \Omega(U,f) \text{ with } r \in (\mathcal{R}.(U)_f)_0 \text{ s.t. for all } (x,\mathfrak{p})\in V,\ s_{x,\mathfrak{p}} \text{ is the image of } r \text{ under } (\mathcal{R}.(U)_f)_0 \longrightarrow ((\mathcal{R}_x)_f)_0 \longrightarrow ((\mathcal{R}_x)_{\mathfrak{p}})_0 . \}$$

Then $\underline{\mathrm{Proj}}(\mathcal{R}.)$ is a locally ringed space and we have a morphism $\pi : \underline{\mathrm{Proj}}(\mathcal{R}.) \longrightarrow X$ of preschemes with $(x,\mathfrak{p}) \longmapsto x$ and

$$\pi^* : \mathcal{O}_X \longrightarrow \pi_* \mathcal{O}_{\mathrm{Proj}(\mathcal{R}.)}$$
$$f \in \mathcal{O}_X(U) \longmapsto (\text{image of } f \text{ under } \mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to (\mathcal{R}_0)_x \to ((\mathcal{R}_x)_{\mathfrak{p}})_0)_{(x,\mathfrak{p})\in\pi^{-1}}$$

The construction is base-local: $\underline{\mathrm{Proj}}(\mathcal{R}.|_U) \cong \pi^{-1}(U)$, and for a graded ring $R.$, $X = \mathrm{Spec}\, R_0$, $\mathcal{R} = \widetilde{R.}$ we get $\mathrm{Proj}(\mathcal{R}.) \cong \mathrm{Proj}(R.)$.

Fact 2.3.1 a) $\underline{\mathrm{Proj}}_X(\mathcal{R}.)$ is a prescheme; in fact $\pi^{-1}(U)$ is a scheme for all aff open $U\subseteq X$

b) π is separated. If $\mathcal{R}.$ is loc. of f.t. as $\mathcal{O}_X$-algebra, then π is of f.t. If X is (locally) Noetherian, $\mathcal{R}.$ of loc. f.t. then $\underline{\mathrm{Proj}}(\mathcal{R})$ (loc) Noeth

c) When $\mathcal{R}.$ is generated by $\mathcal{R}_0$ and $\mathcal{R}_1$ as $\mathcal{O}_X$-algebra, then the $\mathcal{O}(\ell)$ are linebundles. (Sheafification of $\Omega(U,f) \longmapsto (\mathcal{R}.(U)_f)\ell.$)

Moreover

$$\mathcal{O}(k)^{-1} \cong \mathcal{O}(-k) \qquad \mathcal{O}(k)\otimes\mathcal{O}(\ell) \cong \mathcal{O}(k+\ell)$$

We have a canonical morphism $\mathcal{R}_k \longrightarrow \pi_*\mathcal{O}(k)$: $\underset{\mathcal{R}_k(U)}{r} \longmapsto ([r] \in ((\mathcal{R}_x)_{\mathfrak{p}})_k)_{(x,\mathfrak{p})\in\pi^{-1}}$

Remark: Recall the several left-adjoints of f_*:

$$(f^b\mathcal{G})(U) := \lim_{V\supseteq f(U)} \mathcal{G}(V), \quad f^*\mathcal{G} = (f^b\mathcal{G})^{sh}, \quad f^*\mathcal{M} = f^*\mathcal{M} \otimes_{f^*\mathcal{O}_Y} \mathcal{O}_X .$$

Lemma 2.3.1: We have an adjunction relation similar to $\mathrm{Hom}(X,\mathrm{Spec}\, A) \cong \mathrm{Hom}(A, f_*\mathcal{O}_X)$

Let $R.$ be an $\mathbb{N}$-graded ring generated by R_0, R_1, $S = \mathrm{Spec}(R_0)$ and $X \xrightarrow{\xi} S$ an S-prescheme, where $X = \mathrm{Spec}\, A$ is affine.

- Let $\mathfrak{M} = \{ (\varphi,\tau) \mid \varphi : X \longrightarrow \mathrm{Proj}(R.),\ \tau : \varphi^*\mathcal{O}(1)_{\mathrm{Proj}\, R} \cong \mathcal{O}_X \}$.
- Let $\mathfrak{N} = \{ \text{surjective } A\otimes_{R_0} R \xrightarrow{\alpha} A[T] \text{ of graded } A\text{alg. } |T|=1 \}$

Then there is a bijection $\mathfrak{M} \cong \mathfrak{N}$, compatible with restrictions to affine open subsets of X.

Proof: For $(\varphi,\tau)\in\mathfrak{M}$, every $r\in R_k$ defines global section $r \in \mathcal{O}_{\mathrm{Proj}\,R}(k)(\mathrm{Proj}\,R)$ and we put $\alpha(a\otimes r) = a\cdot \tau^{\otimes k}(r)\cdot T^k$

For $\alpha\in\mathfrak{N}$, let φ be $X \cong \mathbb{P}^0_A \cong \mathrm{Proj}(A[T]) \xrightarrow{\alpha_*} \mathrm{Proj}(A\otimes_{R_0} R.) \longrightarrow \mathrm{Proj}(R.)$ and the isomorphism $\tau : \varphi^*\mathcal{O}(1)_{\mathrm{Proj}(R.)} \cong \mathcal{O}(1)_{\mathbb{P}^0_A} \cong \mathcal{O}_X$

Proposition 2.3.1: For S any prescheme, $X \xrightarrow{\xi} S$ an S-prescheme, $\mathcal{L}$ a line-bundle on X and $\mathcal{R}$ a q.c. graded $\mathcal{O}_S$-algebra generated by $\mathcal{R}_0$ and $\mathcal{R}_1$.

Then there is a bijection $\mathfrak{M}(X) \cong \mathfrak{N}(X)$ where

- $\mathfrak{M}(X) = \{ (\varphi,\tau) \mid \varphi : X \to P = \mathrm{Proj}(\mathcal{R}.),\ \tau : \varphi^*\mathcal{O}(1)_P \xrightarrow{\cong} \mathcal{L} \text{ iso} \}$
- $\mathfrak{N}(X) = \{ \text{Graded } \alpha : \xi^*\mathcal{R} \longrightarrow \bigoplus_{k=0}^{\infty} \mathcal{L}^{\otimes k} \text{ of } \mathcal{O}_X\text{-algebras, surjective on stalks} \}$

Cor 2.3.1 In the above situation $\mathrm{Hom}_{S\text{-}Presc}(X, \underline{\mathrm{Proj}}_S(\mathcal{R}))$ is in canonical bijection with the set of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a representative of an isomorphism class of line bundles on X and α a morphism $\alpha : \xi^*\mathcal{R} \longrightarrow \bigoplus_{k=0}^{\infty} \mathcal{L}^{\otimes k}$ of graded $\mathcal{O}_X$-algebras defining surjections on stalks

Cor 2.3.2: With the above assumptions and an S-prescheme $T \xrightarrow{\tau} S$, we have

$$\underline{\mathrm{Proj}}_T(\tau^*\mathcal{R}_\cdot) \xrightarrow{\cong} (\underline{\mathrm{Proj}}_S(\mathcal{R}_\cdot)) \times_S T$$

via the projection $\underline{\mathrm{Proj}}_T(\tau^*\mathcal{R}_\cdot) \to T$ and $\underline{\mathrm{Proj}}_T(\tau^*\mathcal{R}_\cdot) \longrightarrow \underline{\mathrm{Proj}}_S(\mathcal{R}_\cdot)$ defin. using prop 2.3.1 applied to

$$(\tau\pi_T)^*\mathcal{R}_k = \pi_T^*\tau^*\mathcal{R}_k \xrightarrow{\varepsilon_k} \mathcal{O}_{P_T}(k)$$

For an $\mathbb{N}$-graded q.c. $\mathcal{O}_S$-algebra $\mathcal{R}_\cdot$ and $\mathcal{L}$ a line bundle on S, set

$$\mathcal{R}_\cdot^{(\mathcal{L})} = \bigoplus_{k=0}^{\infty} \mathcal{R}_k \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes k} \qquad (r\otimes\lambda)\cdot(s\otimes\mu) = (rs)\otimes(\lambda\otimes\mu).$$

Cor 2.3.3: There is a canonical isomorphism $\underline{\mathrm{Proj}}(\mathcal{R}_\cdot^{(\mathcal{L})}) \xrightarrow[i]{\cong} \underline{\mathrm{Proj}}(\mathcal{R}_\cdot)$ (write $P^{(\mathcal{L})} := \underline{\mathrm{Proj}}(\mathcal{R}_\cdot^{(\mathcal{L})})$) with an isomorphism

$$i^*(\mathcal{O}(1)_P) \otimes \pi^*_{P^{(\mathcal{L})}}\mathcal{L} \xrightarrow[\tau]{\cong} \mathcal{O}(1)_{P^{(\mathcal{L})}}$$

such that $\pi^*_{P^{(\mathcal{L})}}(\mathcal{R}_1 \otimes \mathcal{L}) \cong \pi^*_{P^{(\mathcal{L})}}(\mathcal{R}_1) \otimes \pi^*_{P^{(\mathcal{L})}}(\mathcal{L}) \longrightarrow i^*\mathcal{O}(1)_P \otimes \pi^*_{P^{(\mathcal{L})}}\mathcal{L}$ $\xrightarrow[\cong]{\tau} \mathcal{O}(1)_{P^{(\mathcal{L})}}$ coincides with $\pi^*_{P^{(\mathcal{L})}}(\mathcal{R}_1 \otimes \mathcal{L}) \longrightarrow \mathcal{O}(1)_{P^{(\mathcal{L})}}$

Proposition 2.3.2: Let $\mathcal{R}_\cdot$ and $\mathcal{S}_\cdot$ be $\mathbb{N}$-graded $\mathcal{O}_S$-algebras as in prop 2.3.1, $\mathcal{T}_k := \mathcal{R}_k \otimes \mathcal{S}_k$ and $\mathcal{T} = \bigoplus_{k=0}^{\infty} \mathcal{T}_k$. Then

$$\underline{\mathrm{Proj}}(\mathcal{R}_\cdot) \times_S \underline{\mathrm{Proj}}(\mathcal{S}_\cdot) \xrightarrow{\cong} \underline{\mathrm{Proj}}(\mathcal{T}_\cdot)$$

are canonically isomorphic.

Proof: One defines it using prop 2.3.1 and then checks the universal property by showing that any epi $\tau^*\mathcal{T}_\cdot \xrightarrow{\alpha} \bigoplus_{k=0}^{\infty} \mathcal{L}^{\otimes k} T^k$ can be written as tensor product of $\tau^*\mathcal{R}_\cdot \xrightarrow{\beta} \bigoplus_{k=0}^{\infty} \mathcal{M}^k T^k$, $\tau^*\mathcal{S}_\cdot \xrightarrow{\gamma} \bigoplus_{k=0}^{\infty} \mathcal{N}^k T^k$, where $\mathcal{L} \cong \mathcal{M} \otimes \mathcal{N}$.

Example 2.3.1: We have a closed embedding $\mathbb{P}^n_R \otimes \mathbb{P}^m_R \longrightarrow \mathbb{P}^{n+m+nm}_R$

Def 2.3.1: A morphism $X \xrightarrow{\xi} S$ of preschemes is projective if there is an $\mathbb{N}$-graded q.c. $\mathcal{O}_S$-algebra $\mathcal{R}_\cdot$ generated by $\mathcal{R}_1$ s.t. $\mathcal{R}_1$ is a loc. f.g. $\mathcal{O}_S$-module and $X \cong \underline{\mathrm{Proj}}(\mathcal{R}_\cdot)$ as preschemes over S.

It is called strongly projective if it factors as $X \xrightarrow{i} \mathbb{P}^n_S \longrightarrow S$, where i is a closed embedding.

Fact 2.3.2: If $X \xrightarrow{\xi} Y$ is (strongly) projective, then the base change $\hat{X} := X \times_Y \hat{Y} \longrightarrow \hat{Y}$ is as well. If $\hat{Y} \xrightarrow{\upsilon} Y$ is also (strongly) projective, the same holds for $\hat{X} \longrightarrow Y$.

Proof: If $X \cong \underline{\mathrm{Proj}}_Y(\mathcal{R}_\cdot)$, then $\hat{X} \cong \underline{\mathrm{Proj}}_{\hat{Y}}(\upsilon^*\mathcal{R}_\cdot)$ by cor. 2.3.2. If also $\hat{Y} \cong \underline{\mathrm{Proj}}_Y(\mathcal{S}_\cdot)$, $\hat{X} = \underline{\mathrm{Proj}}(\mathcal{T}_\cdot)$ as in prop 2.3.2.

For strongly projectivity, a factorization $X \xrightarrow{i} \mathbb{P}^n_Y \to Y$ gives factorization $\hat{X} \xrightarrow{\hat{i}} \mathbb{P}^n_Y \times_Y \hat{Y} \longrightarrow \hat{Y}$ and $\mathbb{P}^n_Y \times_Y \hat{Y} \cong \mathbb{P}^n_{\hat{Y}}$. If $\hat{Y} \longrightarrow Y$ factors over $\mathbb{P}^m_Y$, $\hat{X} \longrightarrow Y$ factors as $\hat{X} \longrightarrow \mathbb{P}^n_Y \times_Y \hat{Y} \longrightarrow \mathbb{P}^n_Y \times_Y \mathbb{P}^m_Y \longrightarrow \mathbb{P}^{n+m+nm}_Y \longrightarrow Y$.

Prop 2.33: If $i: X \to Y$ is a closed immersion, $f: Y \to S$ a (strongly) projective morphism. Then $X \xrightarrow{f \circ i} S$ is also (strongly) projective.

Proof: For f projective, let $Y = \mathrm{Proj}(\mathcal{R}_\cdot)$. Let $\mathcal{J}_X \subseteq \mathcal{O}_Y$ be the sheaf of ideals defining $X \subseteq Y$ and let $\mathcal{I}_n \subseteq \mathcal{R}_n$ be the preimage of $f_* \mathcal{J}_X(n) \subseteq f_* \mathcal{O}_Y(n)$ under $\mathcal{R}_n \to f_* \mathcal{O}(n)$. Let $\mathcal{I} = \bigoplus_{n=0}^{\infty} \mathcal{I}_n$ and put $\tilde{\mathcal{R}} = \mathcal{R}/\mathcal{I}$, $\tilde{X} = \mathrm{Proj}_S(\tilde{\mathcal{R}})$. $\mathcal{R}_\cdot \to \tilde{\mathcal{R}}_\cdot$ defines a closed embedding $\tilde{X} \hookrightarrow Y$. We want to show that $X = \tilde{X}$ as closed subpreschemes of Y. Let $\mathcal{I}_X$ be the sheaf of ideals defining $\tilde{X}$. We want $\mathcal{J}_X = \mathcal{I}_X$. This can be checked locally, so assume $S = \mathrm{Spec}(A)$, $Y = \mathrm{Proj}(R)$, $\tilde{X} = \mathrm{Proj}(R/I)$ for a graded A-algebra R and ideal $I \subseteq R$. On open sets $U = Y \setminus V(\lambda)$ for $\lambda \in R_d$, we have $\mathcal{O}_Y(U) = (R_\lambda)_0$, $\mathcal{I}_X(U) = (I_\lambda)_0$, so we need to show that $\mathcal{J}_X(U) = (I_\lambda)_0$. We have $(I_\lambda)_0 \subseteq \mathcal{J}_X(U)$ by definition of I. For the converse:

Lemma 2.33: In the situation of before, let $d > 0$. Let $\lambda \in R_d$ and $U = Y \setminus V(\lambda)$. If $g \in \mathcal{J}_X(n)(U)$, then there is some $k \in \mathbb{N}$ and $\gamma \in I_{dk+n}$ such that $g = \gamma \lambda^{-k}$, i.e. $g \in (I_\lambda)_n$.

Proof: Apply lemma 2.2.2(b) to the q.c. $\mathcal{O}_Y$-module $\mathcal{J}_X$ and the line bundle $\mathcal{O}(d)$, to see that $g\lambda^k$ extends to $\gamma_k \in \mathcal{J}_X(dk+n)(Y)$ for large k.

Cor 2.34 a) Every strongly projective morphism is projective.
b) If S has an ample line bundle, any projective $X \to S$ is strongly proj.

Proof: a) Apply prop 2.3.3. to $X \to \mathbb{P}^n_S \to S$, where $\mathbb{P}^n_S \to S$ is projective.
b) Let $X = \mathrm{Proj}_S(\mathcal{R})$. If $\mathcal{L}$ ample line bundle, then $\mathcal{R}_1 \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes k}$ is generated by global sections for $k \gg 0$. By cor 2.3.3 we may replace $\mathcal{R}_k$ by $\mathcal{R}^{(\mathcal{L}^{\otimes k})}$ and assume $\mathcal{R}_1$ to be generated by its global sections (finitely many, as $\mathcal{R}_1$ locally f.g. and S q.c.) say $\xi_0, \dots, \xi_n$. Then $\mathcal{O}_S[X_0, \dots, X_n] \to \mathcal{R}$: $X_i \mapsto \xi_i$ defines a closed embedding $X \to \mathbb{P}^n_S$ of S-preschemes, so $X \to S$ is strongly projective.

Prop 2.34: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are strongly projective, then gf is. If f and g are only projective, then gf is projective if in addition Z is q.c. and q.s.

Proof: Given closed embeddings $X \to \mathbb{P}^m_Y$, $Y \to \mathbb{P}^n_Z$, we get
$$X \to \mathbb{P}^m_Y \to \mathbb{P}^m_{\mathbb{P}^n_Z} \cong \mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z \to \mathbb{P}^{n+m+nm}_Z \to Z.$$

Example 2.32: Let $\mathcal{A}$ be a q.c. $\mathcal{O}_S$-algebra and $\mathcal{R}$ the graded $\mathcal{O}_S$-algebra defined by
$$\mathcal{R}_k = \begin{cases} \mathcal{O}_S \cdot T^0 & k = 0 \\ \mathcal{A} \cdot T^k & k > 0. \end{cases}$$
Then $\mathrm{Proj}_S(\mathcal{R}) \cong \mathrm{Spec}_S(\mathcal{A})$, as $V(T) = \emptyset$ and $\mathrm{Proj}(\mathcal{R}) \setminus V(T) \cong \mathrm{Spec}((\mathcal{R}_T)_0)$

Cor 2.35: Finite morphisms are projective.

Theorem 4: Let $f: X \longrightarrow S$ be a projective morphism of locally Noetherian preschemes and M a coherent $\mathcal{O}_X$-module. Then each $R^k f_* M$ is coherent.

Proof: W.m.a. $S = \operatorname{Spec} A$, $X = \operatorname{Proj} R$, where A is Noetherian, R a graded A-algebra generated by R_0 and R_1, with $R_0 = A$, R_1 f.g. A-module. If $\xi_0, \dots, \xi_n \in R_1$ are generators, they give closed embedding $X \xhookrightarrow{i} \mathbb{P}^n_A$ and then $f = pi$, where $p: \mathbb{P}^n_A \longrightarrow \operatorname{Spec}(A)$. By cor 1.6.1, $R^k f_* M = R^k p_*(i_* M)$ and since $i_* M$ is coherent, $\cancel{H}$ $H^k(\mathbb{P}^n_A, i_* M)$ is f.g. by theorem 3, so $R^k p_*(i_* M)$ coherent.

Proposition 5: If $X \xrightarrow{f} Y \xrightarrow{g} S$ are s.t. g is separated and gf (strongly) projective, then f is (strongly) projective.

Lemma 2.3.4: Let $\mathcal{C}$ be a class of separated morphisms stable under base change and s.t. if $f \in \mathcal{C}$, i closed embedding, then $fi \in \mathcal{C}$. Then $gf \in \mathcal{C}$ with g separated implies $f \in \mathcal{C}$.

2.4 Proper morphisms

Definition 2.4.1: We call $X \xrightarrow{f} Y$ with Y locally Noetherian proper if it is of finite type, separated and universally closed (all base changes closed).

Prop 2.4.1: The class of proper morphisms is base-local, stable under composition and base change and if gf is proper, g separated, then f is proper.

Prop. 2.4.2: $\operatorname{Proj}(\mathcal{R}) \longrightarrow X$ is proper for a loc. f.g. graded $\mathcal{O}_X$-algebra $\mathcal{R}$. In particular, projective morphisms are proper.

Proof: W.m.a. $X = \operatorname{Spec} A$, R an A-algebra of finite type. Suffices to show that ~~Spec~~ $\operatorname{Proj}(R) \longrightarrow \operatorname{Spec}(A)$ has closed image. Let R be generated by $R_0, \dots, R_d$ and let $I_i = \operatorname{Ann}_A(\bigoplus_{j \geq i} R_j) = \bigcap_{j=i}^{i+d} \operatorname{Ann}_A(R_j) \subseteq A$, $I_\infty = \bigcup_{i=0}^{\infty} I_i$. Remains to show: image is $V(I_\infty) = \bigcap_{i=0}^{\infty} V(I_i)$. For $\mathfrak{p} \in \operatorname{Spec}(A)$, $\mathfrak{p} \in V(I_\infty)$ iff $R_j \otimes k(\mathfrak{p}) \neq 0$ for infinitely many j iff $\operatorname{Proj}(R \otimes k(\mathfrak{p})) \neq 0$, and this is the fibre of $\mathfrak{p}$.

Theorem 5: Let $X \xrightarrow{f} Y$ be a proper morphism of locally Noeth. preschemes. If M is a coherent $\mathcal{O}_X$-module, all $R^p f_* M$ are coherent $\mathcal{O}_Y$-modules.

Cor 2.4.1: If A Noetherian, X a proper A-scheme and M a coherent $\mathcal{O}_X$-module, then $H^p(X, M)$ is a f.g. A-module.

Theorem 6: Let $X \xrightarrow{f} \operatorname{Spec} A$ be proper, $\mathcal{L}$ a line bundle on X. T.f.a.e.

(a) $\mathcal{L}$ is ample

(b) Some power $\mathcal{L}^{\otimes k}$ of $\mathcal{L}$ is very ample

[Here $\mathcal{L}'$ is very ample if it is generated by global sections $\ell_0, \dots, \ell_n$ and the morphism $X \longrightarrow \mathbb{P}^n_A$ defined by

$$\mathcal{O}_X[T_0, \dots, T_n] \longrightarrow \bigoplus_{j=0}^{\infty} \mathcal{L}'^{\otimes j} : T_i \longmapsto \ell_i$$

is a closed embedding.]

(c) Some power $\mathcal{L}^{\otimes k}$ of $\mathcal{L}$ is generated by global sections $\ell_0, \dots, \ell_n$ and the above morphism $X \longrightarrow \mathbb{P}^n_A$ is finite (equivalently: affine)

(d) If M is a coherent $\mathcal{O}_X$-module, then for $p > 0$

$$H^p(X, M \otimes \mathcal{L}^{\otimes k}) = 0$$

for large enough k.

Remark: For proving Theorem 5, one uses Chow's lemma to reduce to theorem 4.

3. Cohomology of curves (With restructering of the chapter)

For a q.c. regular curve C, let C_1 be the set of closed points of C and let $\mathrm{Div}(C)$ be the free abelian group generated by C_1. For $D \in \mathrm{Div}(C)$, define

$$\mathcal{O}_C(D)(U) := \{ f \in K = \mathcal{O}_{C,\eta} \mid v_x(f) \geq -D(x) \text{ for all } x \in U \cap C_1 \}$$

giving a <u>line bundle $\mathcal{O}_C(D)$</u> on C. All line bundles on C have this form.

Recall that we defined the first Chern class $c_1 : \mathrm{Pic}(C) \longrightarrow H^1(C, \Omega_{C/k})$ by

$$\mathrm{Pic}(C) := \left\{ \begin{matrix} \text{iso. classes of} \\ \text{line bundles on } C \end{matrix} \right\} \xrightarrow{\cong} \left\{ \begin{matrix} \text{iso classes of} \\ \mathcal{O}_C^*\text{-torseurs} \end{matrix} \right\} \xrightarrow{\cong} H^1(C, \mathcal{O}_C^*) \xrightarrow{\mathrm{dlog}} H^1(C, \Omega_{C/k})$$

3.1 Formulation of the results

Let k be algebraically closed. All schemes are over $\mathrm{Spec}\, k$ and $\Omega_X^1 := \Omega_{X/k}$.

<u>Theorem 7 (Serre Duality)</u>: Let $C \longrightarrow \mathrm{Spec}\, k$ be a proper regular curve.

(a) There is a homomorphism $\deg : \mathrm{Pic}(C) \longrightarrow \mathbb{Z}$ such that

$$\deg \mathcal{O}_C(D) = \deg(D) := \sum_{x \in C_1} D(x)$$

(b) There is a unique isomorphism $T : H^1(C, \Omega_C) \xrightarrow{\cong} k$ such that

$$(2) \qquad \begin{array}{ccc} \mathrm{Pic}(C) & \xrightarrow{c_1} & H^1(C, \Omega_C) \\ \downarrow{\scriptstyle \deg} & & \downarrow{\scriptstyle T}\,{\scriptstyle \cong} \\ \mathbb{Z} & \xrightarrow{\cdot 1} & k \end{array}$$

(c) For every vector bundle $\mathcal{V}$ on C, there is a non-degenerate pairing

$$H^0(C, \mathcal{V}) \times H^1(C, \Omega_C \otimes \mathcal{V}^*) \xrightarrow{m} H^1(C, \Omega_C) \xrightarrow{\cong} k$$

with $\mathcal{V}^* := \mathcal{H}om(\mathcal{V}, \mathcal{O}_C)$, in other words

$$\begin{array}{ccc} H^0(C, \mathcal{V}) & \xrightarrow{\cong} & \mathrm{Hom}(H^1(C, \Omega_C \otimes \mathcal{V}^*), H^1(C, \Omega_C)) \\ v & \longmapsto & (\text{map induced by } \ell \otimes w \mapsto w(v) \cdot \ell) \end{array}$$

Notice that $\Omega_C \otimes \mathcal{V}^* \cong \mathcal{H}om(\mathcal{V}, \Omega_C)$.

<u>Theorem 8 (Riemann-Roch)</u>: Let $g := \dim_k \Omega_C(C)$ be the <u>genus</u> of C. Then for any vector bundle $\mathcal{V}$ on C

$$(3) \qquad \chi(C, \mathcal{V}) := \dim_k \mathcal{V}(C) - \dim_k H^1(C, \mathcal{V}) = \deg(\det \mathcal{V}) + \dim(\mathcal{V})(1-g).$$

Here $\det \mathcal{V} = \Lambda^{\dim(\mathcal{V})} \mathcal{V}$ is a line bundle with universal alternating d-linear form $\mathcal{V}^{\dim(\mathcal{V})} \longrightarrow \det \mathcal{V}$.

Note that by Serre duality $\dim_k H^1(C, \mathcal{V}) = \dim_k H^0(C, \Omega_C \otimes \mathcal{V}^*)$ $= \dim_k \mathrm{Hom}(\mathcal{V}, \Omega_C)$, so we can rewrite (3) as

$$(4) \qquad \dim_k \mathcal{V}(C) - \dim_k \mathrm{Hom}(\mathcal{V}, \Omega_C) = \deg(\det(\mathcal{V})) + \dim(\mathcal{V})(1-g).$$

<u>Remark 3.1.1</u>: For line bundles, we get $\chi(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$. If K is the <u>canonical divisor</u> with $\mathcal{O}_C(K) \cong \Omega_C$, then (4) becomes for $\mathcal{V} = \mathcal{O}_C(D)$

$$\dim_k \mathcal{O}_C(D) - \dim_k \mathrm{Hom}(\mathcal{O}_C(D), \mathcal{O}_C(K)) = \deg D + 1 - g$$

so with $\ell(D) := \dim_k \mathcal{O}_C(D)$

$$(5) \qquad \ell(D) - \ell(K - D) = \deg D + 1 - g$$

<u>Fact 3.1.1</u>: $\deg \Omega_C = \deg K = 2g - 2$

<u>Cor 3.1.1</u>: A divisor D satisfies Riemann's inequality: $\ell(D) \geq \deg D + 1 - g$

3.2 First proofs

We will always consider proper regular integral curves C. (Thus Noetherian and irreducible.) Let η be the generic point and $K := \mathcal{O}_{C,\eta}$ which is a field.

Fact 3.2.1: For divisors $D_1, D_2 \in \mathrm{Div}(C)$, $\mathcal{O}_C(D_1) \otimes \mathcal{O}_C(D_2) \cong \mathcal{O}_C(D_1 + D_2)$.

Proof: Can be checked at stalks of closed points where it follows by definition.

Fact 3.2.2: For Theorem 7 (a) and Theorem 8, it suffices to show that

(6) $\qquad \chi(C, \mathcal{O}_C(D)) = \deg D + \chi(C, \mathcal{O}_C) \qquad$ (all $D \in \mathrm{Div}(C)$)

and

(7) $\qquad \chi(C, \mathcal{O}_C) = 1 - g$

Proof: Theorem 7(a) follows since every line bundle is of the form $\mathcal{O}_C(D)$ and $\deg(\mathcal{O}_C(D)) := \chi(C, \mathcal{O}_C(D)) - \chi(C, \mathcal{O}_C)$ does the job by (6).
For Theorem 8, we have the case of line bundles, and the general case will follow if we prove additivity of both sides in (3), since each vector bundle has a filtration with one-dimensional filtration quotients.
But if $0 \to \mathcal{V} \to \mathcal{U} \to \mathcal{W} \to 0$ is a s.e.s of vector bundles, we have indeed $\chi(C, \mathcal{U}) = \chi(C, \mathcal{V}) + \chi(C, \mathcal{W})$, $\dim(\mathcal{U}) = \dim(\mathcal{V}) + \dim(\mathcal{W})$ and $\det \mathcal{U} \cong \det \mathcal{V} \otimes \det \mathcal{W}$.

Fact 3.2.3: For all $D \in \mathrm{Div}(C)$, (6) holds.

Proof: For $D = \emptyset$ this is trivial, so by induction it remains to show that for $p \in C$ a closed point

$$\chi(C, \mathcal{O}_C(D \pm p)) = \chi(C, \mathcal{O}_C(D)) \pm 1$$

By symmetry the case "$-p$" is sufficient. Write $\mathcal{L} = \mathcal{O}_C(D)$, $\mathcal{L}(\pm p) := \mathcal{L} \otimes \mathcal{O}_C(\pm p)$.
Let $\mathrm{Spec}\, k \cong \{p\} \xrightarrow{i} C$ be the closed embedding. Let π be a uniformizer of the DVR $\mathcal{O}_{C,p}$ ($v_p(\pi) = 1$, i.e. p generates maximal ideal) and let $\lambda \in \mathcal{L}_p$ be a generator. We get a s.e.s.

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \xrightarrow{\rho} i_* \mathcal{O}_{\mathrm{Spec}\, k} \to 0$$

where $\mathcal{L}(-p) \to \mathcal{L}$ comes from $\mathcal{O}_C(-p) \hookrightarrow \mathcal{O}_C$ and ρ sends $\ell \in \mathcal{L}(V)$ (with $p \in V$) to $(\ell_p/\lambda \bmod \pi) \in \mathcal{O}_{C,p}/\pi\mathcal{O}_{C,p} \cong k$. Then the claim follows by additivity of $\chi(C, -)$ and $\chi(C, i_*\mathcal{O}_{\mathrm{Spec}\, k}) = \chi(\mathrm{Spec}\, k, \mathcal{O}_{\mathrm{Spec}\, k}) = 1$.

Cor 3.2.1: For $C = \mathbb{P}^1_k$, $\mathrm{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$ and $\mathcal{O}_{\mathbb{P}^1_k}(D) \cong \mathcal{O}(\deg(D))$.

Proof: First part is an exercise. For the second, we have $\deg(\mathcal{O}(1)) = \pm 1$ as $\mathcal{O}(1)$ generates $\mathrm{Pic}(\mathbb{P}^1_k)$. But $\deg(\mathcal{O}(1)) = -1$ would give a morphism $\mathcal{O}(1) \to \mathcal{O}(-1)$, which is impossible since $\mathcal{H}om(\mathcal{O}(1), \mathcal{O}(-1)) \cong \mathcal{O}(-2)$ has no global sections.

Fact 3.2.4: Theorem 7 and Theorem 8 hold for $C = \mathbb{P}^1_k$.

Proof: For theorem 7(b), it suffices to prove $\Omega_{\mathbb{P}^1_k} \cong \mathcal{O}(-2)$ and to show that $c_1(\mathcal{O}(1))$ has non-zero image in $H^1(C, \mathcal{O}(-2))$.
Assuming this for a moment, theorem 7(c) follows since theorem 2(e) gives the result for $\mathcal{V} = \mathcal{O}(d)$ and by an exercise, every $\mathcal{V}$ is a direct sum of line bundles $\mathcal{O}(d)$. By fact 3.2.2, we now need $\chi(C, \mathcal{O}_C) = 1 - g$, but we have $\dim_k \mathcal{O}_C(C) = \dim_k k = 1$ (k alg. closed) and $\dim_k H^1(C, \mathcal{O}_C) = \dim_k H^0(C, \Omega_C) = g$ by Serre duality of theorem 7(b) and (c).

Define a derivation $d: \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow \mathcal{O}(-2)$ by setting for $f \in \mathcal{O}_{\mathbb{P}^1_k}(U)$

$$d(f) = \begin{cases} \frac{1}{Y}\frac{\partial f}{\partial X} & \text{on } U \setminus V(Y) \\ \frac{1}{X}\frac{\partial f}{\partial Y} & \text{on } U \setminus V(X) \end{cases}$$

which is a universal derivation on both $\mathbb{P}^1_k \setminus V(X)$ and $\mathbb{P}^1_k \setminus V(Y)$, so $\Omega_{\mathbb{P}^1_k} \cong \mathcal{O}(-2)$.
The line bundle $\mathcal{O}(1)$ corresponds to the $\mathcal{O}_{\mathbb{P}^1_k}^*$-torseur $\mathcal{O}(1)^*$, which corresponds to the element $\frac{X}{Y} \in \check{H}^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_k}^*)$, where $U_0 = \mathbb{P}^1_k \setminus V(X)$, $U_1 = \mathbb{P}^1_k \setminus V(Y)$. This gets sent to $d\log(\frac{X}{Y}) = \frac{d(X/Y)}{X/Y} = \frac{1/Y^2}{X/Y} = \frac{1}{XY} \in H^1(\mathbb{P}^1_k, \mathcal{O}(-2))$, which gets sent to $1 \in k$.
So $H^1(\mathbb{P}^1_k, \Omega_{\mathbb{P}^1_k}) \xrightarrow{\cong} k$ and the above diagram commutes by $\deg(\mathcal{O}(1)) = 1$.

3.3 The functor $f^!$

We construct a right adjoint $f^!: \mathcal{O}_D\text{-mod} \longrightarrow \mathcal{O}_C\text{-mod}$ to f_* when $f: C \longrightarrow D$ is a finite separable morphism of regular connected curves of f.t. over $k = \bar{k}$.
Let R be a Dedekind domain, $K = Q(R)$, L/K finite separable field extension, $S \subseteq L$ alg. cl. of R in L and $Tr_{L/K}: L \longrightarrow K$ the trace. We put

$$S^* = \{\sigma \in L \mid Tr_{L/K}(\sigma s) \in R \text{ for all } s \in S\} \subseteq L$$

which is a fractional ideal. Its inverse is the <u>Dedekind different</u>

$$\mathfrak{D}_{S/R} := (S^*)^{-1} = \{\sigma \in L \mid \sigma\tau \in S \text{ for all } \tau \in S^*\}$$

The construction commutes with localization, so
<u>Fact 3.3.1</u>: There is a unique sheaf of ideals $\mathcal{D}_f \subseteq \mathcal{O}_C$ s.t. for all $U \subseteq D$ affine open

$$\mathcal{D}_f(f^{-1}U) = \mathfrak{D}_{\mathcal{O}_C(f^{-1}U)/\mathcal{O}_D(U)}$$

We may view $\mathcal{D}_f^{-1}$ as a subsheaf of $\mathcal{K}_C$. Then $Tr_{L/K}: L \longrightarrow K$ gives

$$Tr: f_*\mathcal{K}_C \longrightarrow \mathcal{K}_D \quad \text{and} \quad Tr_{C/D}: f_*\mathcal{D}_f^{-1} \longrightarrow \mathcal{O}_D$$

and we define $Tr_f = Tr_{C/D}$ as the composition

$$f_*(f^*M \otimes_{\mathcal{O}_C} \mathcal{D}_f^{-1}) \cong M \otimes_{\mathcal{O}_D} f_*\mathcal{D}_f^{-1} \xrightarrow{id \otimes Tr} M \otimes_{\mathcal{O}_D} \mathcal{O}_D \cong M$$

<u>Prop 3.3.1</u>: For every q.c. $\mathcal{O}_C$-module N, there is a bijection

$$\begin{aligned} Hom_{\mathcal{O}_C}(N, f^*M \otimes_{\mathcal{O}_C} \mathcal{D}_f^{-1}) &\xrightarrow{\cong} Hom_{\mathcal{O}_D}(f_*N, M) \\ \varphi &\longmapsto Tr_f \circ f_*(\varphi) \end{aligned}$$

Thus $f^!: Qc(D) \longrightarrow Qc(C): M \longmapsto f^*M \otimes_{\mathcal{O}_C} \mathcal{D}_f^{-1}$ is <u>right adjoint to f_*.</u>
We will now work toward $f^!\Omega_D \cong \Omega_C$, but we need a lot of algebra.
<u>Lemma 3.3.1</u>: Let f as above, $d \in D$ a closed point and $c_1, \dots, c_N$ its preimages.
(a) There is an neighborhood U of d and $\xi \in \mathcal{O}_C(f^{-1}U)$ s.t. $\xi(c_i) \neq \xi(c_j)$ if $i \neq j$ and $\xi - \xi(c_i) \notin \mathfrak{m}_{C,c_i}^2 \subseteq \mathcal{O}_{C,c_i}$ for $1 \le i \le N$.
(b) If ξ has this property, we can shrink U s.t. ξ generates $\mathcal{O}_C(f^{-1}U)$.
<u>Prop 3.3.2</u>: Let R, K, L, S be as in the start of section 3.3 and let $\xi \in S$ generate S as an R-module, with minimum polynomial $p \in R[T]$.
Then $\mathfrak{D}_{S/R} = (p'(\xi))$ is the principal ideal generated by $p'(\xi)$.
<u>Lemma 3.3.2</u>: In this above situation, write $\frac{p(T)}{T-\xi} = b_0 + b_1 T + \dots + b_{n-1}T^{n-1}$.
Then $(\frac{b_i}{p'(\xi)})_{i=0}^{n-1}$ is a base of S/R dual to $\{\xi^i\}_{i=0}^{n-1}$ w.r.t. $(x,y) \longmapsto Tr_{L/K}(x \cdot y)$.
<u>Prop 3.3.3</u>: If $C \xrightarrow{f} D$ is a finite separable morphism between regular curves over $k = \bar{k}$, then $f^*\Omega_D \longrightarrow \Omega_C$ can be extended to a unique isomorphism

$$f^!\Omega_D = f^*\Omega_D \otimes_{\mathcal{O}_C} \mathcal{D}_f^{-1} \xrightarrow{\cong} \Omega_C$$

Prop 3.3.4: For the isomorphism $f^!\Omega_D \cong \Omega_C$, we have the following commutative diagram:

$$\begin{tikzcd}
f_*\mathcal{O}_C^* \arrow[r, "f_*(d\mathrm{clog})"] \arrow[d, "N_{L/K}"] & f_*\Omega_C \cong f_*f^!\Omega_D \arrow[d, "\mathrm{Tr}_f"] \\
\mathcal{O}_D^* \arrow[r, "d\log"] & \Omega_D
\end{tikzcd}$$

where $K := \mathcal{O}_{D,\eta_D}$, $L := \mathcal{O}_{C,\eta_C}$.

Proof: It suffices to prove that for a finite separable field extension L/K, we have for $\ell \in L^*$ that $\mathrm{Tr}_{L/K}(d_{L/k}\log \ell) = d_{K/k}\log N_{L/K}(\ell)$. Let $\sigma_1, \dots, \sigma_n : L \hookrightarrow M$ be the n K-linear embeddings, then

$$d_{K/k}\log(N_{L/K}(\ell)) = \frac{d_{K/k}N_{L/K}(\ell)}{N_{L/K}(\ell)} = \frac{d_{K/k}(\prod_i \sigma_i(\ell))}{\prod \sigma_i(\ell)} = \sum_{i=1}^n \frac{d_{M/k}(\sigma_i(\ell))}{\sigma_i(\ell)}$$
$$= \sum \sigma_i\left(\frac{d_{M/k}(\ell)}{\ell}\right) = \mathrm{Tr}_{L/K}(d_{L/k}\log \ell)$$

3.4 Proof of Serre duality for curves

Again let C always be a proper regular curve over $k = \bar{k}$.

Prop 3.4.1: Let $f \in \mathcal{O}_{C,\eta}$ be a rational function on C with $df \neq 0$ and let P be the set of poles of f. Then the morphism

$$C \setminus P \xrightarrow{f} \mathbb{A}^1 = \mathrm{Spec}\, k[T]$$

defined by $k[T] \to \mathcal{O}_C(C\setminus P) : T \mapsto f$ extends to a unique finite separable morphism

$$C \xrightarrow{f} \mathbb{P}^1_k$$

(Note that such $f \in \mathcal{O}_{C,\eta}$ always exists.)

Proof: As $f^* d_{\mathbb{A}^1/k} T = df \neq 0$, $\Omega_{L/K} = 0$ and L/K is separable. Let N be the set of zeros of f, then $C\setminus N \xrightarrow{1/f} \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ extends $f|_{C\setminus(N\cup P)}$ and we can glue them together. Uniqueness follows from C being separated, $C\setminus P$ dense. One can show that f is affine and thus finite.

Proof of Theorem 7 and Theorem 8

For a proper regular curve C, take $f : C \to \mathbb{P}^1_k$ as in prop 3.4.1. By the previous results and the fact that we have Serre duality for $\mathbb{P}^1_k$, we get

$$H^1(C,\mathcal{V})^* \cong H^1(\mathbb{P}^1_k, f_*\mathcal{V})^* \cong \mathrm{Hom}_{\mathbb{P}^1_k}(f_*\mathcal{V}, \Omega_{\mathbb{P}^1_k})$$
$$\cong \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, f^!\Omega_{\mathbb{P}^1_k}) \cong \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \Omega_C) = H^0(C, \Omega_C \otimes \mathcal{V}^*).$$

In particular for $\mathcal{V} = \Omega_C$, $H^1(C,\Omega_C) \cong H^0(C, \Omega_C \otimes \Omega_C^*)^* \cong \mathcal{O}_C(C)^* \cong k$.

The isomorphism satisfies theorem 7(b) by the commuting square

$$\begin{tikzcd}
\mathrm{Pic}(C) \arrow[rr, "c_1"] \arrow[dr, "\deg"] \arrow[dd] & & H^1(C,\Omega_C) \arrow[dr, "\cong"] \arrow[dd, "\mathrm{Tr}_f"] & \\
& \mathbb{Z} \arrow[rr, dotted] & & k \\
\mathrm{Pic}(\mathbb{P}^1_k) \arrow[rr, "c_1"] \arrow[ur, "\deg_{\mathbb{P}^1}"] & & H^1(D,\Omega_{\mathbb{P}^1_k}) \arrow[ur, "\cong"] &
\end{tikzcd}$$

by prop 3.3.4. So Theorem 7(b),(c) hold and 7(a), 8 follow as before.

Remark: One can now prove the Hurewicz formula:

$$2g_C - 2 = \sum_{c \in C_1} v_c(\mathcal{D}_f) + \deg(f)(2g_D - 2)$$

where $\deg(f)$ is the rank of $f_*\mathcal{O}_C$ as an $\mathcal{O}_D$-vector bundle.

4. Cohomology and base change

4.1 Base change by a flat morphism

Fact 4.1.1: Let A be a ring, B a flat A-algebra and $X \to \operatorname{Spec} A$ a q.c. $\operatorname{Spec} A$-scheme. Let $Y := X \times_{\operatorname{Spec} A} \operatorname{Spec} B$, with $Y \xrightarrow{\pi} X$ the projection. Then for every q.c. $\mathcal{O}_X$-module $\mathcal{M}$, we have

$$H^\bullet(Y, \pi^*\mathcal{M}) \cong H^\bullet(X, \mathcal{M}) \otimes_A B$$

Proof: For $\operatorname{Spec} R = U \subseteq X$ an open affine subset, write $\mathcal{M}|_U = \tilde{M}$ for some R-module M. $U \times_{\operatorname{Spec} A} \operatorname{Spec} B = \operatorname{Spec}(R \otimes_A B)$ is also affine and $\pi^*\mathcal{M} = (M \otimes_A B)^\sim$. So $\pi^*\mathcal{M}(\pi^{-1}(U)) \cong \mathcal{M}(U) \otimes_A B$. Applying this to an open affine cover $\mathcal{U} : X = \bigcup_{i \in I} U_i$ of X gives

$$\check{C}^\bullet(\pi^{-1}\mathcal{U}, \pi^*\mathcal{M}) \cong \check{C}^\bullet(\mathcal{U}, \mathcal{M}) \otimes_A B$$

which proves the result by exactness of $- \otimes_A B$.

4.2 The theorem about formal functions

Let $X \xrightarrow{f} \operatorname{Spec} R$ be a proper morphism and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, where R is a Noetherian ring. Let $X_n = X \times_{\operatorname{Spec} R} \operatorname{Spec}(R/\mathfrak{m}^{n+1})$ with closed embedding $X_n \xrightarrow{i_n} X$.

Theorem 9: In the above case

$$\widehat{H^p(X, \mathcal{M})} \cong \varprojlim_{n \in \mathbb{N}} H^p(X_n, i_n^*\mathcal{M})$$

where $\wedge$ denotes completion w.r.t. $\mathfrak{m}$.

4.3 Base change for a flat proper morphism

$$H^p(X \times_{\operatorname{Spec} A} \operatorname{Spec} B, \mathcal{V}) \cong H^p(\mathcal{G}^\bullet(\mathcal{V}) \otimes_A B)$$